Heat Conduction in Composite Regions of Analytical Solution of Boundary Value Problems with Arbitrary Convection Boundary Conditions

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Abstract

An analytical solution is presented for non-homogeneous, one dimensional, transient heat conduction problem in composite region, such as multilayer slab, cylinders and spheres. With arbitrary convection conditions on both outer surfaces. The method of solution is based on separation of variables and on orthogonal expansion of functions over multilayer regions.

Keywords: *analytical method, heat conduction, heat generation within solid, non-homogeneous.*

1. Introduction

In modern engineering applications, multilayer components are extensively used due to the added advantage of combining physical, mechanical and thermal properties of different material, composite regions such as multilayer slabs, cylinders and spheres are often encountered in thermal and thermodynamic systems. Solution of transient heat conduction problems in such regions, has numerous applications in thermal sciences, including nuclear and space technology Related solution procedures may be analytical [1]-[3], numerical [4]-[6] or approximate [3], [7], [8].

The usual analytical solutions of transient heat conduction problems in composite bodies include the Laplace transform technique [2], the adjoint solution method [9], the orthogonal expansion technique [10], the method of separation of variables and orthogonal expansion over multilayer regions [11], etc. Although the above method may, in principle, be applied to problems with various kinds and combinations of boundary conditions, there are cases, like the one considered here in which the general solution procedure should be considerably modified.

In the present study, the case of convection boundary conditions on both boundaries is examined, i.e. it is considered that hear is exchanged between the outer boundary surfaces of the composite region and the surrounding fluids, the temperatures of which vary arbitrary with time. Although this problem is encountered very often in practice, little attention has received thus far. The proposed solution procedure of the above problem is based on the method of separation of variables and of orthogonal expansion of functions over multilayer regions, a general description of which may be found in [3],[11]. The imposition of convection boundary conditions on both outer surfaces with time dependence surrounding temperatures, which are different in each boundary, makes the problem non homogeneous heat generation within solid.

2. The non homogeneous problem

One dimensional heat conduction is considered in a composite region consisting of m parallel layers slabs, concentric cylinders or spheres. In perfect thermal contact, the thermal properties are discontinuous at the interface between the layers but they remain uniform within each layer. The temperature distribution for each layer is prescribed for time t=0, and convection conditions are imposed for t>0 at the outer boundary surfaces $x=x_1$ and $x=x_{m+1}$.

Using the one dimensional Laplace differential operator with p=0,1,2 for plates, cylinders and spheres respectively.

$$\nabla^2 = \frac{1}{x^p}\frac{\partial}{\partial x}\left(x^p \frac{\partial}{\partial x}\right) \tag{1}$$

The problem may be expressed by differential equations

$$\propto_i \nabla^2 T_i(x,t) + \frac{\propto_i}{k_i} g_i(x,t) = \frac{\partial T_i(x,t)}{\partial x}, \tag{2}$$

t >0, $x_i \le x \le x_{i+1}$, i=1,2,3,……m

Defining: $T_i(x,t)$ =temperature of the i[th] layer in $x_i \leq x \leq x_{i+1}$,
$C_i, k_i, \propto_i, \rho_i$=specific heat , thermal conductivity , thermal diffusivity, density respectively of the i[th] layer.
And x_i and x_{i+1} denotes the coordinates of the i[th] layer surfaces

The boundary conditions at the outer surfaces $x=x_1$ and $x=x_{m+1}$ and at the interfaces $x=x_{i+1}$ are expressed by equations.

$$h_1[T_1(x,t) - T_{s1}(t)] = k_1 \frac{\partial T_1(x,t)}{\partial x}, \quad x=x_1, \qquad (3)$$

$$T_i(x,t) = T_{i+1}(x,t), \quad X=X_{i+1}, \; i=1,2,3,...,m-1, \qquad (4)$$

$$k_i \frac{\partial T_i(x,t)}{\partial x} = k_{i+1} \frac{\partial T_{i+1}(x,t)}{\partial x}, \quad X=X_{i+1}, \; i=1,2,3,...,m-1 \qquad (5)$$

$$h_m[T_m(x,t) - T_{sm}(t)] = -k_m \frac{\partial T_m(x,t)}{\partial x}, \quad x=x_{m+1} \qquad (6)$$

Where h_1 and h_m are the heat transfer coefficients at the outer surfaces x_1 and x_{m+1} respectively , $T_{S1}(t)$ and $T_{sm}(t)$ are the corresponding surrounding temperatures, and k_i is the thermal conductivity of the i[th] layer .

The initial condition is expressed as

$$T_i(x,0)=F_i(x), \quad x_i \leq x \leq x_{i+1}, \; i=1,2,3,\ldots,m, \qquad (7)$$

Where $F_i(x)$ are given functions.

The heat conduction problem described by eqs (2)-(7) is nonhomogeneous because, aithough the interfaces conditions (4) and (5) are homogeneous , the convection boundary conditions at the outer surface (3) and (6) are nonhomogeneous as they contain the terms T_{S1} and T_{sm} respectively . and also differential equation (2) is nonhomogeneous . Hence the problem is nonhomogeneous

3 Homogenization of the problem

The convection boundary conditions may be homogenized by introducing a new dependent variable defined as

$$Z_i(x,t)=T_i(x,t)-q_i(x,t), \quad x_i \leq x \leq x_{i+1}, \; i=1,2,3,\ldots,m \qquad (8)$$

Where

$$q_1(x,t) = \frac{(x_2 - x)^2 h_1 T_{s1}(t)}{(x_2 - x_1)(h_1 x_2 - h_1 x_1 + 2k_1)} \qquad (9)$$

$$q_i(x,t) = 0, \quad i=2,3,4,\ldots,m-1. \qquad (10)$$

$$q_m(x,t) = \frac{(x - x_m)^2 h_m T_{sm}(t)}{(x_{m+1} - x_m)(h_m x_{m+1} - h_m x_m + 2k_m)} \qquad (11)$$

Using Eq.(8), the heat conduction differential equation (2)becomes

$$\frac{\partial Z_i(x,t)}{\partial t} = \alpha_i \nabla^2 Z_i(x,t) + \alpha_i \nabla^2 q_i(x,t) + \frac{\alpha_i}{k_i} g_i(x,t) - \frac{\partial q_i(x,t)}{\partial t}$$

$$t > 0, \; x_i \leq x \leq x_{i+1} \qquad (12)$$

the boundary conditions (3)-(6) are transformed to

$$h_1 Z_1(x,t) = k_1 \frac{\partial Z_1(x,t)}{\partial x} \qquad (13)$$

$$Z_i(x_{i+1},t) = Z_{i+1}(x_{i+1},t), \quad i=1,2,3,\ldots m-1 \qquad (14)$$

$$k_i \frac{\partial Z_i(x_{i+1},t)}{\partial x} = k_{i+1} \frac{\partial Z_{i+1}(x_{i+1},t)}{\partial x}, \quad i=1,2,3\ldots m-1 \qquad (15)$$

$$h_m Z_m(x_{m+1},t) = -k_m \frac{\partial Z_m(x_{m+1},t)}{\partial x \partial} \qquad (16)$$

And initial condition (7) becomes

$$Z_i(x,0) = F_i(x) - q_i(x,0) \equiv f_i(x),$$

$$x_i \leq x \leq x_{i+1}, \; i=1,2,3,\ldots m \qquad (17)$$

4 Method of solution

Assuming separation of the variable [3], the solution is expressed in the form

$$Z_i(x,t) = \sum_{n=1}^{\infty} X_{in}(x)\Gamma_n(t), \qquad (18)$$

$$x_i \leq x \leq x_{i+1}, \; i=1,2,3,\ldots m$$

Where function $X_{in}(x)$ and $\Gamma_n(t)$ are determined below.

4.1 Calculation of $X_{in}(x)$

Eigenfuctions $X_{in}(x)$ satisfy the following eigenvalue problem [3]:

$$\propto_i \nabla^2 X_{in}(x) + \beta_n{}^2 X_{in}(x) = 0 \qquad (19)$$

$$x_i \leq x \leq x_{i+1}, \; i=1,2,3,\ldots,m$$

With the boundary conditions

$$h_1 X_{1n}(x_1) = k_1 \frac{dX_{in}(x_1)}{dx} \qquad (20)$$

$$X_{in}(x_{i+1}) = X_{i+1,n}(x_{i+1}) \quad i=1,2,3\ldots\ldots m-1 \qquad (21)$$

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$$k_i \frac{dX_{in}(x_{i+1})}{dx} = k_{i+1}\frac{dX_{i+1,n}(x_{i+1})}{dx}, \quad i=1,2,3\ldots\ldots m-1 \quad (22)$$

$$h_m X_{mn}(x_{m+1}) = -k_m \frac{dX_{mn}(x_{m+1})}{dx} \quad (23)$$

Where β_n are the eigenvalues.

The solution of Eq. (19) may be expressed as

$$X_{in}(x) = A_{in}\emptyset_{in}(x) + B_{in}\varphi_{in}(x) \quad (24)$$

Where

i) **for plates**

$$\emptyset_{in}(x) = \cos\left(\frac{\beta_n x}{\sqrt{\alpha_i}}\right), \quad \varphi_{in}(x) = \sin\left(\frac{\beta_n x}{\sqrt{\alpha_i}}\right) \quad (25)$$

ii) for cylinders

$$\emptyset_{in}(x) = J_0\left(\frac{\beta_n x}{\sqrt{\alpha_i}}\right), \quad \varphi_{in}(x) = Y_0\left(\frac{\beta_n x}{\sqrt{\alpha_i}}\right) \quad (26)$$

iii) for spheres

$$\emptyset_{in}(x) = \frac{1}{x}\sin\left(\frac{\beta_n x}{\sqrt{\alpha_i}}\right), \quad \varphi_{in}(x) = \frac{1}{x}\cos\left(\frac{\beta_n x}{\sqrt{\alpha_i}}\right) \quad (27)$$

And A_{in} , B_{in} are constants determined by solving the following equation set, which results by substitution of Eq. (24) into the boundary conditions (20)-(23)

$$[h_1\emptyset_{in}(x_1) - k_1\emptyset'_{1n}(x_1)]A_{1n} + [h_1\varphi_{1n}(x_1) - k_1\varphi'_{1n}(x_1)]B_{in} = 0 \quad (28)$$

$$\emptyset_{in}(x_{i+1})A_{in} + \varphi_{1n}(x_{i+1})B_{in} - \emptyset_{i+1,n}(x_{i+1})A_{i+1,n} - \varphi_{i+1,n}(x_{i+1})B_{i+1,n} = 0$$
$$i=1,2,3\ldots\ldots m-1 \quad (29)$$

$$k_i\emptyset'_{in}(x_{i+1})A_{in} + k_i\varphi'_{in}(x_{i+1})B_{in} - k_{i+1}\emptyset'_{i+1,n}(x_{i+1})A_{i+1,n} - k_{i+1}\varphi'_{i+1,n}(x_{i+1})B_{i+1,n} = 0$$
$$i=1,2,3\ldots\ldots m-1 \quad (30)$$

$$[h_m\emptyset_{mn}(x_{m+1}) + k_m\emptyset'_{mn}(x_{m+1})]A_{mn} + [h_m\varphi_{mn}(x_{m+1}) + k_m\varphi'_{mn}(x_{m+1})]B_{mn} = 0 \quad (31)$$

Symbols $\emptyset'$ and φ' denotes derivatives of $\emptyset$ and φ with respect to x.

The above homogeneous set of 2m equations will have nonzero solution if the determinant of the coefficients is set to zero,

$$B=0 \quad (32)$$

The positive roots $\beta_1<\beta_2<\ldots<\beta_n$ of the above equation are the eigenvalues . For each eigenvalue β_n solution of the homogeneous set of Eqs.(28)-(31) gives the corresponding 2m constants A_{1n} , B_{1n}, A_{2n} , B_{2n},……… A_{mn}, B_{mn}. One of which should have been determined arbitrarily .

4.2 Calculation of $\Gamma_n(t)$

By orthogonal property of the eigenfunction $X_{in}(x)$ over the entire range of m layers [3], [8]

$$\sum_{i=1}^{m}\frac{k_i}{\alpha_i}\int_{x_i}^{x_{i+1}} X_{in}(x)X_{in'}(x)x^p\,dx = \begin{cases}0 & for\ n \neq n' \\ const. & for\ n = n'\end{cases} \quad (33)$$

Assuming the heat generating function $g_i(x,t)$procedure described in [3],function $f_i(x)$, $\frac{\partial q_i(x,t)}{\partial t}$ and $\alpha_i\,\nabla^2 q_i(x,t)$ are expressed in terms of the eigenfunction $X_{in}(x)$as.

$$\frac{\alpha_i}{k_i}g_i(x,t) = \sum_{n=1}^{\infty} g_n^*(t)X_{in}(x) \quad i=1,2,3,\ldots m \quad (34)$$

$$f_i(x) = \sum_{n=1}^{\infty} f_n^* X_{in}(x) \quad i=1,2,3\ldots m \quad (35)$$

Expand unity in the form

$$1 = \sum_{n=1}^{\infty} I_n^* X_{in}(x) \quad i=1,2,3,\ldots\ldots m \quad (36)$$

$$\frac{\partial q_i(x,t)}{\partial t} = \sum_{n=1}^{\infty} V_n^*(t)X_{in}(x) \quad i=1,2,3,\ldots m \quad (37)$$

$$\alpha_i\,\nabla^2 q_i(x,t) = \sum_{n=1}^{\infty} I_n^*(t)X_{in} \quad i=1,2,3,\ldots\ldots m \quad (38)$$

The unknown function $g_n^*(t)$, f_n^*, I_n^*, $V_n^*(t)$ in the above expansions are determined by multiplying both side of Eqs (34),(35),(36),(37)and(38) by

$$\frac{k_i}{\alpha_i}X_{in}(x).x^p$$

And integrating with respect to x from x_i to x_{i+1} summing up the equalities over all values of I and making use of the orthogonality conditions given by equation (33).

$$g_n^*(t) = \frac{\sum_{i=1}^{m}\int_{x_i}^{x_{i+1}} g_i(x,t)X_{in}(x)x^p\,dx}{N} \quad (39)$$

$$f_n^* = \frac{\sum_{i=1}^{m}\frac{k_i}{\alpha_i}\int_{x_i}^{x_{i+1}} f_i(x)X_{in}(x)x^p\,dx}{N} \quad (40)$$

$$I_n^* = \frac{\sum_{i=1}^{m}\frac{k_i}{\alpha_i}\int_{x_i}^{x_{i+1}} X_{in}(x)x^p\,dx}{N} \quad (41)$$

$$V_n^*(t) = \frac{\sum_{i=1}^{m} \int_{x_i}^{x_{i+1}} \frac{\partial q_i(x,t)}{\partial t} X_{in}(x) x^p \, dx}{N} \qquad (42)$$

$$N = \sum_{i=1}^{m} \frac{k_i}{\alpha_i} \int_{x_i}^{x_{i+1}} X^2{}_{in}(x) x^p \, dx \qquad (43)$$

For the outer layer i=1 and i=m substituting of Eqs(18),(34),(37)and (38) into Eq.(12)

$$\sum_{n=1}^{\infty} \frac{d\,\Gamma_n(t)}{dt} X_{in}(x) = \sum_{n=1}^{\infty} \alpha_i \Gamma_n(t) \nabla^2 X_{in}(x) + \sum_{n=1}^{\infty} I_n^*(t) X_{in}(x) + \sum_{n=1}^{\infty} g_n^*(t) X_{in}(x) - \sum_{n=1}^{\infty} V_n^*(t) X_{in}(x) \qquad (44)$$

$$\sum_{n=1}^{\infty} \left[\frac{d\,\Gamma_n(t)}{dt} + V_n^*(t) + \beta_n^2 \Gamma_n(t) - I_n^*(t) - g_n^*(t) \right] X_{in}(x) = 0 \qquad (45)$$

Fro which the following differential equation for the calculation of $\Gamma_n(t)$ is obtain

$$\frac{d\,\Gamma_n(t)}{dt} + \beta_n^2 \Gamma_n(t) = I_n^*(t) + g_n^*(t) - V_n^*(t) \qquad (46)$$

The initial condition for Eq.(46) is for t=0

$$Z_i(x,0) = f_i(x) = \sum_{n=1}^{\infty} X_{in}(x) \Gamma_n(0) = \sum_{n=1}^{\infty} X_{in}(x) f_n^* \qquad (47)$$

Integration of differential equation (46) with initial condition

$$\Gamma_n(t) = e^{-\beta_n^2 t} \left[f_n^* + \int_0^t [I_n^*(t) + g_n^*(t) - V_n^*(t)] e^{\beta_n^2 t} dt \right] \qquad (48)$$

For i=1 and i=m

For the intermediate layers For i=2,3,4,……..m-1

$$\sum_{n=1}^{\infty} \alpha_i \Gamma_n(t) \nabla^2 X_{in}(x) + \sum_{n=1}^{\infty} g_n^*(t) X_{in}(x) = \sum_{n=1}^{\infty} \frac{d\,\Gamma_n(t)}{dt} X_{in}(x) \qquad (49)$$

$$\sum_{n=1}^{\infty} \left[\frac{d\,\Gamma_n(t)}{dt} + \beta_n^2 \Gamma_n(t) - g_n^*(t) \right] X_{in}(x) = 0 \qquad (50)$$

Or

$$\frac{d\,\Gamma_n(t)}{dt} + \beta_n^2 \Gamma_n(t) = g_n^*(t) \qquad (51)$$

Integrating of differential equation with initial condition is

$$\Gamma_n(t) = e^{-\beta_n^2 t} \left[f_n^* + \int_0^t [g_n^*(t)] e^{\beta_n^2 t} dt \right]$$

for i=2,3…m-1(52)

4.3 Final expression of the solution

Substitution of Eq.(48),(52) into Eq.(18) then the final expression of the solution
For the layer i=1 and i=m,

$$Z_i(x,t) = \sum_{n=1}^{\infty} X_{in}(x) e^{-\beta_n^2 t} \left[f_n^* + \int_0^t [I_n^*(t) + g_n^*(t) - V_n^*(t)] e^{\beta_n^2 t} dt \right]$$

$$x_i \le x \le x_{i+1} \qquad (53)$$

And for the intermediate layers i=2,3,…..m-1

$$Z_i(x,t) = \sum_{n=1}^{\infty} X_{in}(x) e^{-\beta_n^2 t} \left[f_n^* + \int_0^t [g_n^*(t)] e^{\beta_n^2 t} dt \right] \qquad (54)$$

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