Fractional Fourier Transform of Tempered Boehmians

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Abstract

Tempered Boehmians are introduced as a natural extension of tempered distributions. For this class of Boehmians it is possible to define an extension of the Fractional Fourier transforms. The Fractional Fourier transform of tempered Boehmian is a distribution. Distributions, which are transforms of Boehmians, are characterized and an inversion theorem is proved.

Keywords: *Fractional Fourier transforms, Boehmians, convolution quotients.*

1. **Introduction**.

Boehmians were introduced in [4] as a generalization of both regular operators of Boehme (a subclass of Mikusinski operators) [2] and Schwartz distributions. They have the algebraic character of convolution quotients similar to Mikusinski operators. At the same time, there is no restriction on the support of Boehmians, which is present in the definition of Mikusinski operators. A Schwartz distribution is regular operator if and only if its support is bounded on the left. On the other hand, all Schwartz distributions, as well as Beurling or Roumieu ultradistributions, are Boehmians.

The space of Boehmians is defined by an abstract algebraic construction, which is a generalization of the construction of the field of quotients. The construction applied to different function spaces yields various spaces of generalized functions (see [5], [6], [7], [8] and [9]). If the function space is the space of Lebesgue integrable functions on $\Re^{N}$, the obtained space of Boehmians consists of the so-called integrable Boehmians. It is possible to define the Fractional Fourier transform for integrable Boehmians [7]. The Fractional Fourier transform of an integrable Boehmian is a continuous function. This extension of the Fractional Fourier has desirable properties. Since there are integrable Boehmians, which are not tempered distributions, this extension allows us to use the Fractional Fourier transform in some cases where the theory of distributions cannot be used. On the other hand, there are tempered distributions which are not integrable Boehmians. For example, the distributional Fractional Fourier transform can be applied to polynomials, which are not integrable Boehmians.

In the paper we define a new extension of the Fractional Fourier transform. First we define tempered Boehmians, which include all tempered distributions and other generalized functions. Then we define the Fractional Fourier transform of a tempered Boehmian, The extension has the usual properties. The Fractional Fourier transform of a tempered Boehmians is a distribution. We characterize distributions, which are transforms of tempered Boehmians and prove an inversion theorem.

2. **Tempered Boehmians**: Denote by T the space of slowly increasing infinitely differential complex –valued functions on $\Re^{N}$ (f is called slowly increasing if there exists a polynomial p such that $\left|f(x)\right| \le p(x)$ for all $x \in \Re^{N}$). By $D(\Re^{N})$, or simply D , we denote the space of all infinitely differentiable complex-valued functions on $\Re^{N}$ with compact support. By a delta sequence we mean a sequence of real-valued functions $\delta_1.\delta_2.... \in D$ such that

(i) $\int_R \delta_n(x)d(x) = 1, \qquad \forall n \in N,$

(ii) $\int_R \left|\delta_n(x)\right| d(x) \le M \quad \forall n,$ for some M > 0,

(iii) For every ε > 0 there exists $n_0 \in \mathrm{N}$ such that $\delta_n(x) = 0$ for $|x| \ge \varepsilon$ and $n > n_0$.

By the convolution $f * g$ of two functions f and g we mean the function defined, as $(f * g)(x) = \int_{\Re^N} f(u)g(x-u)du$ whenever the integral exists.

A pair of sequence (f_n, φ_n) is called a quotient of sequence, denoted for short by f_n/φ_n , if $f_n \in T$ for all $n \in \mathrm{N}$, $\{\varphi_n\}$ is a delta sequence, and $f_n * \varphi_m = f_m * \varphi_n$ for all $m,n \in \mathrm{N}$. Two quotients of sequences f_n/φ_n and g_n/γ_n are equivalent if $f_n * \gamma_m = g_m * \varphi_n$ for all $m,n \in \mathrm{N}$. The equivalence class of f_n/φ_n is denoted by $\left[f_n/\varphi_n\right]$and the space of all equivalence classes quotients of sequences in denoted by

B_T. Elements of B_T are called tempered Boehmians. B_T is a complex vector with the addition and multiplication by a scalar defined as follows:

$$[f_n/\varphi_n]+[g_n/\gamma_n]=[(f_n*\gamma_n+g_n*\varphi_n)/(\varphi_n*\gamma_n)]$$

and $\alpha[f_n/\varphi_n]=[\alpha f_n/\varphi_n]$. Let $F=[f_n/\varphi_n]\in B_T$. Partial derivatives of F are defined as follows:

$$\frac{\partial F}{\partial x_m}=\left[\left(\frac{\partial F_n}{\partial x_m}*\varphi_n\right)\Big/(\varphi_n*\varphi_n)\right]$$

Note that $(\partial f_n/\partial x_m)*\varphi_n$ is a slowly increasing function for every $n\in\mathrm{N}$ and that $((\partial f_n/\partial x_m)*\varphi_n)/(\varphi_n*\varphi_n)$ is a quotient of sequences. Thus partial derivatives of tempered Boehmians are tempered Boehmians.

Let f be an infinitely differentiable complex valued function on $\Re^{\mathrm{N}}$.

If $\sup_{|\alpha|\le m}\sup_{x\in\Re^{\mathrm{N}}}(1+x_1^2+....+x_N^2)^m\left|\frac{\partial^{|\alpha|}f(x)}{\partial x^\alpha}\right|<\infty$ for every nonnegative integer m, then f is called rapidly decreasing. In the above we use the following notation: $\alpha=(\alpha_1......\alpha_N)$ is a multi-index. α_n are nonnegative integers. $|\alpha|=\alpha_1+....+\alpha_N$ and $\frac{\partial^{|\alpha|}}{\partial x^\alpha}=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}.....\partial x_N^{\alpha_N}}$.

Let $\tau(\Re^{\mathrm{N}})$ or simply τ, denote the space of all rapidly decreasing functions on $\Re^{\mathrm{N}}$. A tempered Boehmian $F=[f_n/\varphi_n]$ is called a rapidly decreasing Boehmians if $F=[f_n/\varphi_n]\in B_T$ and $f_n\in\tau$ for all $n\in\mathrm{N}$, The space of all rapidly decreasing Boehmians is denoted by B_τ. If $F=[f_n/\varphi_n]\in B_T$ and $G=[g_n/\gamma_n]\in B_\tau$, then the convolution $F*G$ can be defined as $F*G=[(f_n*g_n)/(\varphi_n*\gamma_n)]$. It is easy to see that $F*G\in B_T$.

3. **The Fractional Fourier Transform of Tempered Boehmians**: In this section we defined the Fractional Fourier transform of tempered Boehmians. It is important to distinguish between convolution quotients and the usual quotients. Although it should always be clear form the context which one is meant, we use f/φ to denote a convolution quotient and $\frac{f}{\varphi}$ to denote a usual quotient.

Let $f\in T$. The Fractional Fourier transform of f, denoted by $\hat{f}$, is the distribution **defined:** The fractional Fourier transforms (fractional FT) R^α is an extension of the ordinary Fourier transforms and depends on a parameter α. For 0 < α ≤ л / 2, fractional FT reduces to the ordinary Fourier transforms [1].

An entire function $\hat{\phi}(t)$ on ¢ is the fractional FT R^α on $L^1(R)$ is defined by

$$\hat{\phi}(t)=R^\alpha\phi(t)=\int_{-\infty}^{\infty}\phi(x)K_\alpha(x,t)dm(x),$$

where the kernel,

$$K_\alpha(x,t)=(2\pi i\sin\alpha)^{-1/2}\exp\left(i\alpha/2\right)\exp\left(\frac{i}{2\sin\alpha}((x^2+t^2)\cos\alpha-2xt)\right)$$

x is restricted to compact set.

If A and C are constant such that,

$$|\hat{\phi}(t)|\le C\exp\{-M(|t|/A)+H_K(t)\}$$

where,

$H_K(t)=\sup_{x\in k}(-C_{2\alpha}\,\mathrm{Im}((x^2+t^2)\cos\alpha-2xt))$, is the support function.

Suppose that f is an ultra - distribution with compact support in R. Hence

$$[R^\alpha f(x)](t)=\hat{f}(t)=F_\alpha(t)=\langle f(x),K_\alpha(x,t)\rangle$$

defines an entire function on ¢, which we call the fractional Fourier Transforms of f.

We can define the inversion fractional FT, $[R^\alpha]^{-1}$ of tempered distribution by duality.

$$\langle[R^\alpha]^{-1}f,\phi\rangle=\langle f,[R^\alpha]^{-1}\phi\rangle,\ f\in D',\phi\in D \quad .$$

$f=[R^\alpha]^{-1}[R^\alpha]f=R^\alpha([R^\alpha]^{-1}f,\ f\in D'$, holds.

Also $[R^\alpha]^{-1}[f]=C_\alpha\ \ (2\pi)^n R^\alpha[\breve{f}],\ f\in D'$ where $\breve{f}(\xi)=f(-\xi,-\alpha);C\alpha=\frac{-2C_{2\alpha}e^{i\alpha}}{i(C_{1\alpha})^2}$.

Inversion Fractional Fourier transform is also denoted by $(f\hat{\delta}_n)^{\smile}$ of Fractional Fourier transforms of $f\hat{\delta}_n$.

Theorem 1. If $[f_n/\varphi_n]\in B_T$, then the sequence $\{\hat{f}_n\}$ converges in D'. Moreover, if $[f_n/\varphi_n]=[g_n/\gamma_n]\in B_T$, then $\{\hat{f}_n\}$and $\{\hat{g}_n\}$ converge to the same limit.

Proof: Let $\varphi \in D$ and let $k \in N$ be such that $\hat{\varphi}_k > 0$ on the support of φ. Since $f_n * \varphi_m = f_m * \varphi_n$ for all $m, n \in \mathrm{N}$, we have $\hat{f}_n \hat{\varphi}_m = \hat{f}_m \hat{\varphi}_n$. Thus

$$\hat{f}_n(\varphi) = \hat{f}_n(\varphi\hat{\varphi}_n) = \left(\hat{f}_n\hat{\varphi}_k\right)\left(\frac{\varphi}{\hat{\varphi}_k}\right) = \left(\hat{f}_k\hat{\varphi}_n\right)\left(\frac{\varphi}{\hat{\varphi}_k}\right) = \hat{f}_k\left(\frac{\varphi\hat{\varphi}_n}{\hat{\varphi}_k}\right)$$

.

Since the sequence $\left\{\frac{\varphi\hat{\varphi}_n}{\hat{\varphi}_k}\right\}$ converges to $\frac{\varphi}{\hat{\varphi}_k}$ in D, the sequence $\left\{\hat{f}_n(\varphi)\right\}$ converges. This proves that the sequence $\left\{\hat{f}_n\right\}$ converges in D'. Now assume that $[f_n/\varphi_n] = [g_n/\gamma_n] \in B_T$. Define $h_n = \left\{ f_{\frac{n+1}{2}} * \gamma_{\frac{n+1}{2}} \right.$ if n is odd and $h_n = \left\{ g_{\frac{n}{2}} * \varphi_{\frac{n}{2}} \right.$ if n is even. And $\delta_n = \left\{ \varphi_{\frac{n+1}{2}} * \gamma_{\frac{n+1}{2}} \right.$ if n is odd, $\delta_n = \left\{ \varphi_{\frac{n}{2}} * \gamma_{\frac{n}{2}} \right.$ if n is even.

Then $[h_n/\delta_n] = [f_n/\varphi_n] = [g_n/\gamma_n]$. By the first part of this proof, the sequence $\left\{\hat{h}_n\right\}$ converges in D'. Moreover,

$$\lim_{n\to\infty} \hat{h}_{2n-1}(\varphi) = \lim_{n\to\infty}\left(f_n * \gamma_n\right)^{\cap}(\varphi) = \lim_{n\to\infty}\left(\hat{f}_n\hat{\gamma}_n\right)$$

$$= \lim_{n\to\infty} \hat{f}_n(\hat{\gamma}_n\varphi) = \lim_{n\to\infty} \hat{f}_n(\varphi)$$

.

Thus $\left\{\hat{h}_n\right\}$ and $\left\{\hat{f}_n\right\}$ have the same limit. Similarly, $\left\{\hat{h}_n\right\}$ and $\left\{\hat{g}_n\right\}$ must have the same limit. This completes the proof.

Definition 2. Let $F = [f_n/\varphi_n] \in B_T$. By the Fractional Fourier transform of F, denoted by $\hat{F}$, we mean the limit of the sequence $\left\{\hat{f}_n\right\}$ in D'.

The defined Fractional Fourier transforms in thus a mapping form B_T into D'. It is clearly a linear mapping. Below we prove some other properties of the Fractional Fourier transforms.

Theorem 3. Let $F = [f_n/\varphi_n] \in B_T$. Then

$$\left(\frac{\partial F}{\partial x_m}\right)^{\cap} = ix_m\hat{F}.$$

Proof:

$$\left(\frac{\partial F}{\partial x_m}\right)^{\cap} = \left[\left(\frac{\partial f_n}{\partial x_m} * \varphi_n\right) \Big/ \left(\varphi_n * \varphi_n\right)\right]^{\cap} = \lim_{n\to\infty}\left(\frac{\partial f_n}{\partial x_m} * \varphi_n\right)^{\cap}$$

$$= \lim_{n\to\infty} ix_m \hat{f}_n\hat{\varphi}_n = ix_m\hat{F}$$

. The last equality follows from the $[f_n/\varphi_n] = [(f_n * \varphi_n)/(\varphi_n * \varphi_n)]$ and from Theorem 1.

Lemma 4. If $G \in B_\tau$, then $\hat{G}$ is an infinity differentiable function.

Proof. Let $G = [g_n/\gamma_n] \in B_T$ and let U be a bounded open subset of $\mathfrak{R}^N$. Then there exists $m \in N$ such that $\hat{\gamma}_m > 0$ on U and we have

$$\hat{G} = \lim_{n\to\infty} \hat{g}_n = \lim_{n\to\infty} \frac{\hat{g}_n\hat{\gamma}_m}{\hat{\gamma}_m} = \lim_{n\to\infty} \frac{\hat{g}_m\hat{\gamma}_n}{\hat{\gamma}_m} = \frac{\hat{g}_m}{\hat{\gamma}_m}\lim_{n\to\infty}\hat{\gamma}_n = \frac{\hat{g}_m}{\hat{\gamma}_m}$$

on U. Since $\hat{g}_m, \hat{\gamma}_m \in \tau$ and $\hat{\gamma}_m > 0$ on U, $\hat{G}$ is infinitely differentiable on U.

Theorem 5. If $F \in B_\tau$ and $G \in B_\tau$ then $(F * G)^{\cap} = \hat{F}\hat{G}.$

Proof. Let $F = [f_n/\varphi_n] \in B_T$ and $G = [g_n/\gamma_n] \in B_\tau$. If $\varphi \in D$, then there exists $m \in N$ such that $\hat{\gamma}_m > 0$ on the support of φ and we have

$$(F * G)^{\cap}(\varphi) = \lim_{n\to\infty}(f_n * g_n)^{\cap}(\varphi) = \lim_{n\to\infty}\left(\hat{f}_n\hat{g}_n\right)(\varphi) = \lim_{n\to\infty}\hat{f}_n(\hat{g}_n\varphi) = \lim_{n\to\infty}\hat{f}_n\left(\frac{\hat{\gamma}_m\hat{g}_n\varphi}{\hat{\gamma}_m}\right)$$

$$\lim_{n\to\infty}\hat{f}_n\left(\frac{\hat{\gamma}_n\hat{g}_m\varphi}{\hat{\gamma}_m}\right) = \lim_{n\to\infty}\hat{f}_n\left(\frac{\hat{g}_m}{\hat{\gamma}_m}\varphi\hat{\gamma}_n\right) = \lim_{n\to\infty}\hat{f}_n\left(\hat{G}\varphi\hat{\gamma}_n\right) = \hat{G}\lim_{n\to\infty}\hat{f}_n(\varphi\hat{\gamma}_n) = \hat{G}\lim_{n\to\infty}\left(\hat{f}_n\hat{\hat{\gamma}}_n\right)(\varphi)$$

$$= \hat{G}\lim_{n\to\infty}\left(f_n * \gamma_n\right)^{\cap}(\varphi) = \left(\hat{F}\hat{G}\right)(\varphi)$$

=

The last equality follows form the fact that $[f_n/\varphi_n] = [(f_n * \gamma_n)/(\varphi_n * \gamma_n)]$ and form Theorem 1.

Lemma 6. If $F = [f_n/\varphi_n] \in B_T$, then $\hat{F}\hat{\varphi}_m = \hat{f}_m$ for all $m \in N$.

Proof. Let $\varphi \in D.$ Then

$$\left(\hat{F}\hat{\varphi}_m\right)(\varphi) = \hat{F}(\hat{\varphi}_m\varphi) = \lim_{n\to\infty} f_n(\hat{\varphi}_m\varphi) = \lim_{n\to\infty}\left(\hat{f}_n\hat{\varphi}_m\right)(\varphi)$$

$$= \lim_{n\to\infty}\left(\hat{f}_m\hat{\varphi}_n\right)(\varphi) = \lim_{n\to\infty} \hat{f}_m(\hat{\varphi}_n\varphi) = \hat{f}_m(\varphi)$$

.

Theorem 7. A distribution f is the Fourier transform of a tempered Boehmian if and only if there exists a delta sequence $\{\delta_n\}$ such that $f\hat{\delta}_n$ is a tempered distribution for every $n \in N$.

Proof. If $F = [f_n/\varphi_n] \in B_T$ and $f = \hat{F}$, then $f\hat{\phi}_n = \hat{F}\hat{\phi}_n = \hat{f}_n$, by Lemma 6. Thus $f\hat{\phi}_n$ is a tempered distribution.

Now let $f \in D$ and let $\{\delta_n\}$ be a delta sequence such that $f\hat{\delta}_n$ is a tempered distribution for every $n \in N$. Define $F = \left[\left((f\hat{\delta})^{\cup} * \delta_n\right) \big/ (\delta_n * \delta_n)\right]$ where $(f\hat{\delta}_n)^{\cup}$ is the inverse Fractional Fourier transforms of $f\hat{\delta}_n$. (Since $f\hat{\delta}_n$ is a tempered distribution, so is $(f\hat{\delta}_n)^{\cup}$). It is easy to check that F is a tempered Boehmian and that $\hat{F} = f$.

From the above we easily obtain the inversion formula.

Theorem 8. Let F be a tempered Boehmian and $\hat{F} = f$. Then $F = \left[\left((f\hat{\delta})^{\cup} * \delta_n\right) \big/ (\delta_n * \delta_n)\right]$, where $\{\delta_n\}$ is a delta sequence such that $f\hat{\delta}_n$ is a tempered distribution for every $n \in N$.

Note that if $F = [f_n/\varphi_n] \in B_T$ then is the inversion form we can take $\delta_n = \varphi_n$. In this case the formula takes a simpler form: $F = \left[(f\hat{\varphi}_n)^{\cup} / \varphi_n\right]$.

Conclusion

We defined the Fractional Fourier transform of a tempered Boehmian, The extension has the usual properties. The Fractional Fourier transform of a tempered Boehmians is a distribution. We characterized distributions, which are transforms of tempered Boehmians and proved an inversion theorem.

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