CHCS Triangular Prism with Exactly One Right Angled Prismatic Edge

Pranab Kalita[1], Bichitra Kalita[2]

[1] **Department of Mathematics, Gauhati University, Guwahati-781014, Assam, India,**

[2] **Department of Computer Applications (M.C.A), Assam Engineering College, Guwahati-781013, Assam, India,**

Abstract

Andreev's Theorem provides a full description of 3-dimensional compact hyperbolic combinatorial polytope having non-obtuse dihedral angles. Prism is one of such polytope. In this article, with the help of Andreev's Theorem, we have developed some properties of CHCS (compact hyperbolic coxeter symmetric) triangular prisms with exactly one right angled prismatic edge, which are facilitated by the link of graph theory and combinatorics, and it has been found that the number of CHCS triangular prisms with prismatic edges of order

$$[2,3,n\geq 7],\ [2,4,n\geq 5],[2,5,n\geq 5]$$

and $[2,m\geq 6,n\geq 6;m\leq n],\ m,n\in \mathbb{N}$

are three, five, five and zero respectively upto symmetry.

Keywords: Coxeter polytope, Planar graph, Dihedral angles, Prism.

MSC 2010 Codes: 52B10, 20F55, 52A55.

1. Introduction

The angle between two faces of a polytope, measured from perpendiculars to the edge created by the intersection of the planes is called a *dihedral angle*. For a combinatorial polyhedron P with E edges, the space of dihedral angles D_P of all compact hyperbolic polyhedrons that realize P is generally not a convex subset of $\mathbb{R}^E$ [22]. If P has more than four faces, Andreev's Theorem states that the corresponding space D_P obtained by restricting to non-obtuse angles is a convex polytope. Therefore, except tetrahedron, Andreev's Theorem [26] provides a complete characterization of 3-dimensional compact hyperbolic combinatorial polytope having non-obtuse dihedral angles. On the other hand, Roland K. W. Roeder's Theorem [12] provides the classification of compact hyperbolic tetrahedron by restricting to non-obtuse dihedral angles.

A simple polytope P in n-dimensional hyperbolic space H^n is said to be *coxeter*, if the dihedral angles of P are of the form $\frac{\pi}{n}$ where, n is a positive integer ≥ 2.

There is no complete classification of hyperbolic coxeter polytopes. Vinberg proved in [25] that there are no compact hyperbolic coxeter polytopes in H^n when $n\geq 30$. Tumarkin classified the hyperbolic coxeter pyramids in terms of coxeter diagram and John Mcleod generalized it in his article [10]. P. Kalita and B. Kalita [1] found that there are exactly one, four and thirty coxeter Andreev's tetrahedrons having respectively two edges of order $n\geq 6$, one edge of order $n\geq 6$ and no edge of order $n\geq 6$, $n\in \mathbb{N}$ upto symmetry. Again using Roland K. W. Roeder's Theorem, P. Kalita and B. Kalita [2] proved that there are exactly 3 CHC (compact hyperbolic coxeter) tetrahedrons upto symmetry in real projective space. These 3 tetrahedrons can be realized uniquely [11] in Hyperbolic space and these are nothing but the 3 coxeter Andreev's tetrahedrons found by theorems 3.20 and 3.21 in [1]. P. Kalita and B. Kalita [3] also studied about the CHC thin cubes and found that there are exactly 3 such cubes in hyperbolic space upto symmetry.

A triangular prism is a polyhedron made of two triangular base faces and three lateral faces. In this article, with the help of Andreev's Theorem, we have developed some properties of CHCS triangular prism with exactly one right angled prismatic edge. Using graph theory and combinatorics, it has been found that the number of CHCS triangular prisms with prismatic edges of order

$$[2,3,n\geq 7],\ [2,4,n\geq 5],[2,5,n\geq 5]$$

and $[2,m\geq 6,n\geq 6;m\leq n],\ m,n\in \mathbb{N}$

are three, five, five and zero respectively upto symmetry.

The paper is organised as follows:

The section 1 includes introduction. The section 2 includes some basic terminologies from graph theory and geometry. The section 3 focuses some known results from graph theory and geometry. New definitions and results are included in section 4. The section 5 includes the conclusions.

2. Basic Terminologies

There is a strong link between graph theory and geometry. Graph theoretical concepts are used to understand the combinatorial structure of a polytope in geometry. Here we will mention some essential terminologies from graph theory and geometry.

Definition 2.1: A *polytope* is a geometric object with surfaces enclosed by edges that exist in any number of dimensions. A polytope in 2D, 3D and 4D is said to be *polygon*, *polyhedron* (plural polyhedra or polyhedrons) and *polychoron* respectively. The enclosed surfaces are said to be faces. The line of intersection of any two faces is said to be an edge and a point of intersection of three or more edges is called a vertex.

Definition 2.2: Let P be a convex polyhedron. The abstract graph of P is denoted by $G(P)$ and is defined as $G(P)=(V(P),E(P))$, where $V(P)$ is the set of vertices of P and two vertices $x,y\in V(P)$ are adjacent if and only if (x,y) is an edge of P .

Definition 2.3: If the dihedral angle of an edge of a compact hyperbolic polytope is $\frac{\pi}{n}, n$ is a positive number, then n is said to be the *order* of the edge. We define a trivalent vertex to be of order (l,m,n) if the three edges at that vertex are of order l,m,n .

Definition 2.4: A *coxeter* dihedral angle is a dihedral angle of the form $\frac{\pi}{n}$ where, n is a positive integer ≥ 2 . A compact polytope in hyperbolic space with coxeter dihedral angles is called a *compact hyperbolic coxeter polytope*.

Definition 2.5: A cell complex C on S^2 is called trivalent if each vertex is the intersection of three faces.

Definition 2.6: A 3-dimensional combinatorial polytope is a cell complex C on S^2 that satisfies the following conditions:

(a) Each edge of C is the intersection of exactly two faces
(b) A nonempty intersection of two faces is either an edge or a vertex.
(c) Each face is enclosed by not less than 3 edges.

Any trivalent cell complex C on S^2 that satisfies the above three conditions is said to be *abstract polyhedron*.

Definition 2.7: A 3D polytope is called a *simple polytope* if each vertex is the intersection of exactly 3 faces. The 1-skeleton of a polytope is the set of vertices and edges of the polytope. The skeleton of any convex polyhedron is a planar graph and the skeleton of any $k-$ dimensional convex polytope is a $k-$connected graph.

Definition 2.8: A prismatic $k-$ circuit $\Gamma_p(k)$ is a $k-$circuit such that no two edges of C which correspond to edges traversed by $\Gamma_p(k)$ share a common vertex. The edges traversed by a prismatic $k-$circuit $\Gamma_p(k)$ are said to be prismatic edges.

Hyperbolic geometry [2] is difficult to visualize as many of its theorems are contradictory to similar theorems of Euclidean geometry which are very similar to us. Therefore technology has been used in creation of geometric models in Euclidean space to visualize hyperbolic geometry. Hyperbolic [12] planes in these models correspond to Euclidean hemispheres and Euclidean planes that intersect the boundary perpendicularly. Furthermore, these models are correct conformally. That is, the hyperbolic angle between a pair of such intersecting hyperbolic planes is exactly the Euclidean angle between the corresponding spares or planes.

We will concentrate on 3-dimensional compact hyperbolic orbifolds whose base spaces are homeomorphic to a convex polyhedron and whose sides are silvered. The compact hyperbolic polyhedron is simple, therefore, the combinatorial polyhedron of a compact hyperbolic polyhedron can be known from 3-connected planar graph of the polyhedron. In case of compact hyperbolic triangular prism, the corresponding 3-connected planar graph (figure 2.1) has 6 vertices, 9 edges, 5 faces, one prismatic 3-circuit and exactly 3 disjoint edges.

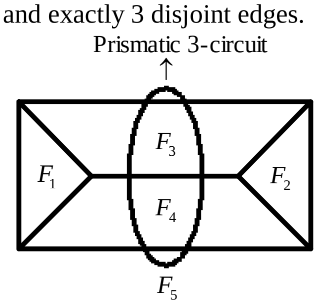

Figure 2.1

3. Known Results

E. M. Andreev provides a complete characterization of 3D compact hyperbolic polytope with non-obtuse dihedral angles in his article [26]. Therefore, Andreev's theorem is a fundamental tool for classification of 3D compact hyperbolic coxeter polytope. Before stating Andreev's Theorem, some already existed results [6] on graph and polytope have been mentioned.

Theorem 3.1: (Blind and Mani) *If P is a convex polyhedron, then the graph $G(P)$ determines the entire combinatorial structure of P. In other words, if two simple polyhedral have isomorphic graphs, then their combinatorial polyhedral are also isomorphic.*

Theorem 3.2: (Ernst Steinitz) *A graph $G(P)$ is a graph of a 3-dimensional polytope P if and only if it is simple, planar and 3-connected.*

Corollary 3.3: *Every 3-connected planar graph can be represented in a plane such that all the edges are straight lines, all the bounded regions determined by these and the union of all the bounded regions are convex polygons.*

Theorem 3.4: (Andreev Theorem, [14, 18]) *Suppose that C is not a tetrahedron and non-obtuse angles $\theta_{ij} \in \left(0, \frac{\pi}{2}\right]$ are given corresponding to each edge $F_{ij} = F_i \cap F_j$ of C, where F_i are the faces of C. Then the following conditions (1) - (4) are necessary and sufficient for the existence of a compact 3-dimensional hyperbolic polyhedron P which realizes C with dihedral angle θ_{ij} at each edge F_{ij}.*

(1) If $F_{ijk} = F_i \cap F_j \cap F_k$ is a vertex of C then $\theta_{ij} + \theta_{jk} + \theta_{ki} > \pi$.

(2) If F_i, F_j, F_k form a prismatic 3-circuit $\Gamma_p(3)$, then $\theta_{ij} + \theta_{jk} + \theta_{ki} < \pi$.

(3) If F_i, F_j, F_k, F_l form a prismatic 4-circuit $\Gamma_p(4)$, then $\theta_{ij} + \theta_{jk} + \theta_{kl} + \theta_{li} < 2\pi$.

(4) If C is a triangular prism with triangular faces F_1 and F_2, then $\theta_{13} + \theta_{14} + \theta_{15} + \theta_{23} + \theta_{24} + \theta_{25} < 3\pi$.

Furthermore, this polyhedron is unique up to isometrics of hyperbolic H^3.

Andreev's restriction to non-obtuse dihedral angles is necessary to ensure that P be convex. A compact hyperbolic polyhedral realizing a given abstract polyhedron may not be convex without the restriction of non-obtuse dihedral angle [18]. Since Coxeter polyhedrons have non-obtuse dihedral angles, Andreev's Theorem provides a complete characterization of 3-dimensional compact hyperbolic Coxeter polyhedra.

Theorem 3.5: ([2]) *In a CHC polytope T, the order of the edges at one vertex is one of the forms:*

$$(2,2,n \geq 2),(2,3,3),(2,3,4),(2,3,5).$$

Theorem 3.6 ([2]): *In a CHC polytope T, if an edge at one vertex is of order $n \geq 6$, then the other two edges must be of order 2.*

Corollary 3.7: *For a compact hyperbolic triangular prism, only the condition (3) of Andreev's theorem is vacuous.*

4. New Definitions and Main Results

Definition 4.1: *A CHCS triangular prism is a CHC triangular prism whose corresponding edges of the base faces are of same order.*

Suppose the orders of the edges of a CHC triangular prism are $n_1, n_2, n_3, n_4, n_5, n_{12}, n_{23}, n_{34}, n_{41}$ as shown in figure 4.1.

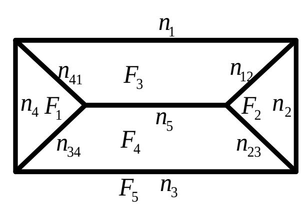

Figure 4.1

In case of CHCS triangular prism, we have $n_{34} = n_{23}, n_4 = n_2$ and $n_{41} = n_{12}$.

Theorem 4.2: *In a compact hyperbolic triangular prism, there exists at most one prismatic edge of order 2.*

Proof: Suppose the orders of the edges of a CHC triangular prism are $n_1, n_2, n_3, n_4, n_5, n_{12}, n_{23}, n_{34}, n_{41}$ as shown in figure 4.2.

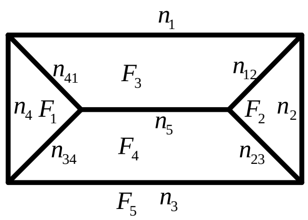

Figure 4.2

Here the orders of the three prismatic edges are n_1, n_5 and n_3.

Suppose at least two out of the three prismatic edges are of order 2 and these are $n_1 = 2$ and $n_5 = 2$. By 2[nd] condition of Andreev's Theorem:

$$\frac{\pi}{n_1} + \frac{\pi}{n_5} + \frac{\pi}{n_3} < \pi$$
$$\frac{1}{n_1} + \frac{1}{n_5} + \frac{1}{n_3} < 1$$
$$\frac{1}{2} + \frac{1}{2} + \frac{1}{n_3} < 1$$
$$1 + \frac{1}{n_3} < 1$$

This is not possible for any positive value for n_3. Therefore, there exists at most one prismatic edge of order 2.

In figure 4.2, the orders of the three prismatic edges formed by the faces F_3, F_4, F_5 are n_1, n_5 and n_3. By 2[nd] condition of Andreev's Theorem:

$$\frac{\pi}{n_1} + \frac{\pi}{n_5} + \frac{\pi}{n_3} < \pi \Rightarrow \frac{1}{n_1} + \frac{1}{n_5} + \frac{1}{n_3} < \pi \qquad \ldots(1)$$

There is an infinite number of solutions to the inequality (1) and by theorem 4.2, there exists at most one prismatic edge of order 2. In this article, we will concentrate only those CHCS triangular prisms whose exactly one prismatic edge is of right angled, that is, of order 2. Without loss of generality, suppose $n_1 = 2$ and hence upto symmetry, the ordered solution (infinite) triplets are:

$$[2, n_5, n_3] = \begin{cases} [2, 3, n \geq 7], \\ [2, 4, n \geq 5], \\ [2, 5, n \geq 5], \\ [2, m \geq 6, n \geq 6; m \leq n] \end{cases}.$$

In our discussion, we treat each of the above triplets as single type only. More clearly, we denote CHCS triangular prisms with prismatic edges of order triplets

$$[2, 3, n \geq 7],\ [2, 4, n \geq 5], [2, 5, n \geq 5]$$
$$\text{and } [2, m \geq 6, n \geq 6; m \leq n],\ m, n \in \mathbb{N}$$

by type P1, P2, P3 and P4 respectively.

Theorem 4.3: *In a compact hyperbolic triangular prism, all the triangular edges cannot be of order 2.*

Proof: Suppose all the triangular edges are of order 2. Therefore from figure 4.1 we have:

$$n_{41} = n_{34} = n_4 = n_{12} = n_{23} = n_2 = 2$$

By 4[th] condition of Andreev's Theorem:

$$\frac{\pi}{n_{41}} + \frac{\pi}{n_{41}} + \frac{\pi}{n_{41}} + \frac{\pi}{n_{41}} + \frac{\pi}{n_{41}} + \frac{\pi}{n_{41}} < 3\pi$$
$$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} < 3$$
$$3 < 3$$

This is not possible. Hence all the triangular edges cannot be of order 2.

Corollary 4.4: *In a compact hyperbolic triangular prism, at least one triangular edge is of order* > 2.

Proof: Straight from Theorem 4.3.

Corollary 4.5: *In a compact hyperbolic triangular prism, at most five triangular edges can be of order 2.*

Proof: Straight from Theorem 4.3.

Theorem 4.6: *In a CHCS triangular prism, if the orders of the prismatic edges are* $[2, 3, n \geq 7]$ *(Type P1) then there exists exactly 3 such prism upto symmetry.*

Proof: Let P be a CHCS triangular prism whose orders of the prismatic edges are $[2, 3, n \geq 7]$ (Type P1). By theorem 3.6, the orders of the edges adjacent to the edge of order $n \geq 7$ must be of order 2. Suppose the rest edge of a base is of order m. Therefore we have the following figure 4.3.

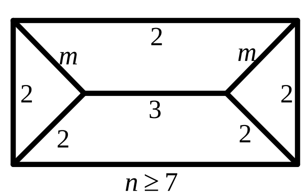

Figure 4.3

ISSN (Online): 2230-7893
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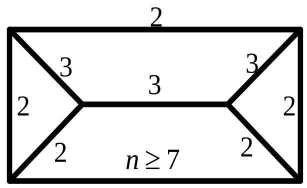
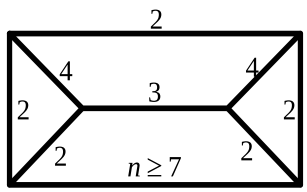
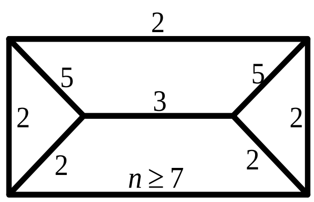
By theorems 3.5 and 4.3, the possible values for $m = 3, 4, 5$ upto symmetry and the respective 3 CHCS triangular prisms of type P1 are:



P1-1



P1-2



P1-3

Figure 4.4

Theorem 4.7: *In a CHCS triangular prism, if the orders of the prismatic edges are* $[2, 4, n \geq 5]$ *(Type P2) then there exists exactly 5 such prism upto symmetry.*

Proof: Let P be a CHCS triangular prism whose orders of the prismatic edges are $[2, 4, n \geq 5]$ (Type P2). For our convenient, we split $[2, 4, n \geq 5]$ into two cases: Case 1: the orders of the prismatic edges are $[2, 4, 5]$**,** Case 2: the orders of the prismatic edges are $[2, 4, n \geq 6]$**.**

Case 1: The orders of the prismatic edges are $[2, 4, 5]$

Suppose, the edges of a base are of orders k, l, m as shown in the following figure.

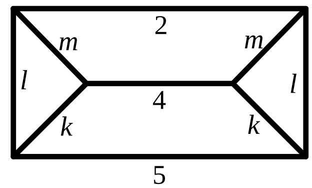


Figure 4.5

By theorems 3.5 and 4.3, the possible values for (k, l, m) can be obtained from the following tree diagram (figure 4.6)

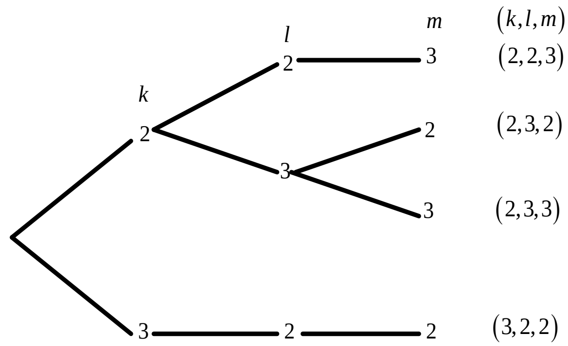
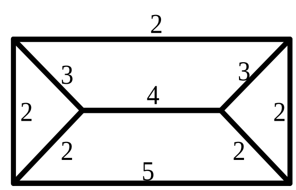
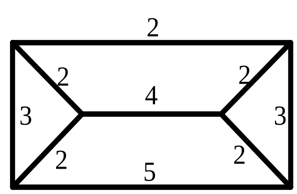


Figure 4.6

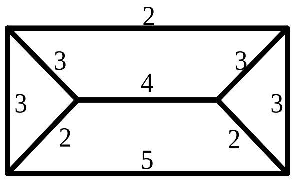
Now the respective 4 CHCS triangular prisms of type P2 upto symmetry are:



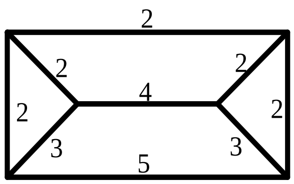
P2-1



P2-2



P2-3



P2-4

Figure 4.7

Case 2: The orders of the prismatic edges are $[2,4,n\geq 6]$

By theorem 3.6, the orders of the edges adjacent to the edge of order $n\geq 6$ must be of order 2. Suppose the rest edge of a base is of order m. Therefore we have the following figure 4.8.

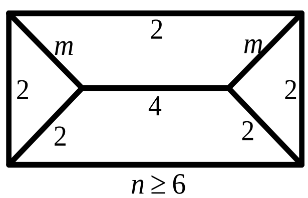
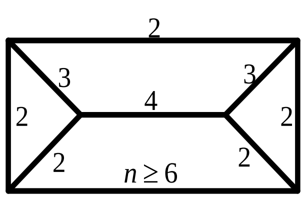


Figure 4.8

By theorems 3.5 and 4.3, the possible value for $m=3$ and the respective 1 CHCS triangular prism upto symmetry is:



P2-5

Figure 4.9

Theorem 4.8: *In a CHCS triangular prism, if the orders of the prismatic edges are* $[2,5,n\geq 5]$ *(Type P3) then there exists exactly 5 such prism upto symmetry.*

Proof: Let P be a CHCS triangular prism whose orders of the prismatic edges are $[2,5,n\geq 5]$ (Type P3). For our convenient, we split $[2,5,n\geq 5]$ into two cases: Case 1: the orders of the prismatic edges are $[2,5,5]$, Case 2: the orders of the prismatic edges are $[2,5,n\geq 6]$.

Case 1: The orders of the prismatic edges are $[2,5,5]$

Suppose, the edges of a base are of orders k,l,m as shown in the following figure 4.10.

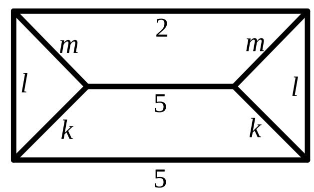
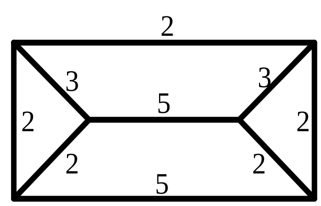


Figure 4.10

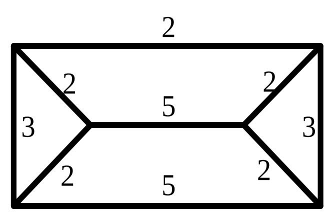
By theorems 3.5 and 4.3, the possible values for

$$(k,l,m)=(2,2,3),(2,3,2),(2,3,3),(3,2,2)$$

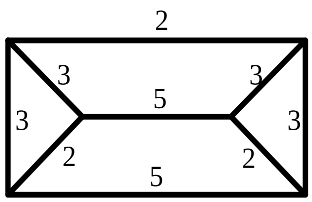
(Refer figure 4.6) and the respective 4 CHCS triangular prisms of type P3 upto symmetry are:



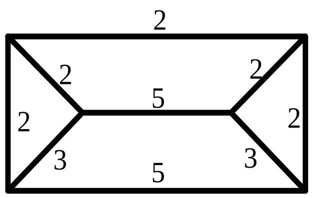
P3-1



P3-2



P3-3



P3-4

Figure 4.11

Case 2: The orders of the prismatic edges are $[2, 5, n \geq 6]$

By theorem 3.6, the orders of the edges adjacent to the edge of order $n \geq 6$ must be of order 2. Suppose the rest edge of a base is of order m. Therefore we have the following figure 4.12.

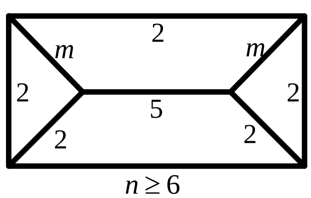
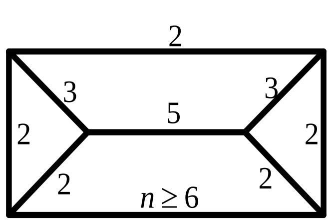


Figure 4.12

By theorems 3.5 and 4.3, the possible value for $m = 3$ and the respective 1 CHCS triangular prism upto symmetry is:



P3-5

Figure 4.13

Theorem 4.9: *There is no CHCS triangular prism with orders of the prismatic edges are* $[2, m \geq 6, n \geq 6; m \leq n]$.

Proof: By theorem 3.6, the orders of the edges adjacent to the edge of order $m \geq 6$ or $n \geq 6$ must be of order 2. Therefore we have the following figure 4.14.

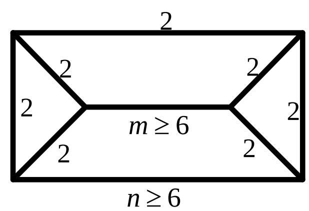


Figure 4.14

Here the orders of all the triangular edges are 2 which are not possible by theorem 4.3. Therefore there is no CHCS triangular prism with orders of the prismatic edges are of the form $[2, m \geq 6, n \geq 6; m \leq n]$.

The CHCS triangular prisms found in theorems 4.6, 4.7 and 4.8 can be realized uniquely [14] in hyperbolic space upto symmetry.

5. Conclusions

In this article, using Andreev's theorem, we have found all the CHCS triangular prisms with exactly one right angled prismatic edge and it has been found that the number of CHCS triangular prisms with prismatic edges of order

$$[2,3,n \geq 7],\ [2,4,n \geq 5], [2,5,n \geq 5]$$

and $[2, m \geq 6, n \geq 6; m \leq n]$, $m, n \in \mathbb{N}$

are three, five, five and zero respectively upto symmetry.

These prisms can be realized uniquely [14] in hyperbolic space upto symmetry. This research can be extended to other compact as well as non-compact hyperbolic polytopes in spaces of different dimensions.

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