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Orderable and Deformable Coxeter Hyperbolic Tetrahedrons with Finite Volume

Dhrubajit Choudhury

Director, HDR Foundation,
Hatigaon, Guwahati, Assam, India

Abstract
In this article, we classified the 3-dimensional orderable and deformable Coxeter hyperbolic tetrahedrons with finite volume. Andreev's states a necessary and sufficient condition for a hyperbolic polyhedron with finite volume. Choi's theorem provides a necessary and sufficient condition for orderable and deformable Coxeter polyhedron. Using graph theory and combinatorics, we find that there are 13 orderable and deformable Coxeter hyperbolic tetrahedrons with finite volume up to symmetry.

Keywords: *Orbifold, Hyperbolic space, Andreev, Orderable, Coxeter, Deformable*

1. Introduction

An *n-dimensional orbifold* is a topological space with a structure based on the quotient space of R^n by a finite group action. An orbifold is called *good* if its universal cover is a manifold. We will concentrate only on good orbifolds.

To give a *hyperbolic structure* on an orbifold, we model it locally by the orbit spaces of finite subgroups of $PO(1,n)$ acting on open subsets of H^n. Similarly, to put a real projective structure on an orbifold, we model it locally by the orbit spaces of finite subgroups of $PGL(n+1,R)$ acting on open subsets of RP^n.

A real projective structure on an orbifold *M* implies that *M* has a universal cover $\tilde{M}$ and the deck transformation group $\pi_1(M)$ acting on $\tilde{M}$ so that $\frac{\tilde{M}}{\pi_1(M)}$ is homeomorphic to *M*.

A *convex* set in RP^n is a convex set in an affine patch. If we use Klein's model of a *n*-dimensional hyperbolic space, then is an open ball in RP^n and $PO(1,n)$ is a subgroup of $PGL(n+1,R)$ preserving H^n. Therefore H^n can be imbedded in an $(n+1)$-dimensional real vector space *V* as an upper part of hyperboloid

$$-x_1^2+x_2^2+...+x_{n+1}^2=-1$$

Hence hyperbolic orbifolds naturally have real projective structures. But a real projective structure of an orbifold may not have hyperbolic structure.

We will concentrate on 3-dimensional compact hyperbolic orbifolds whose base spaces are homeomorphic to a convex polyhedron and whose sides are silvered and each edge is given an order. If the dihedral angle of an edge of a compact hyperbolic polyhedron is $\frac{\pi}{n}$ then we say that the order of the edge is n where n is a positive number.

Definition 1.1.1. Let X be S^3, E^3, or H^3. Let *Isom(X)* denotes the group of isometries of *X*. A *Coxeter polyhedron* is a convex polytope in *X* whose dihedral angles are all integer sub-multiples of π. Let *P* be a 3-dimensional Coxeter polyhedron and Γ be the group generated by the reflections in the faces of *P*. Then Γ is a discrete group of Isom(*X*) and *P* is its fundamental polyhedron. Conversely, every discrete group Γ of Isom(*X*) can be obtained from a Coxeter polyhedron *P* such that *P* is its fundamental polyhedron. The number of faces intersect at vertex is called the *degree* of that vertex. Also the edge order of edges of a Coxeter polyhedron is positive integer.

Definition 1.1.2. Let *P* be a fixed convex polyhedron. Let us assign orders at each edge. Let *e* be the number of edges and e_2 be the numbers of order-two. Let f be number of sides. We remove any vertex of *P* which has more than three edges ending or with orders of the edges ending there is not of the form

$$(2,2,n), n\geq 2, (2,3,3), (2,3,4), (2,3,5)$$

i.e., orders of spherical triangular groups. This make P into an open 3-dimensional orbifold.

Let $\hat{P}$ denote the differential orbifold with sides silvered and the edge orders realized as assigned from P with vertices removed. We say that $\hat{P}$ has a *Coxeter orbifold structure*.

Definition 1.1.3. The deformation space $\hat{P}$ of projective structures on an orbifold $\hat{P}$ is the space of all projective structures on $\hat{P}$ quotient by isotopy group actions of $\hat{P}$.

Definition 1.1.4. We say P is *orderable* if we can order the sides of P so that each sides meets sides meets sides of higher index in less than or equal to 3 edges.

Example 1.1.5. Cube and dodecahedron are not satisfying orderability condition.

Definition 1.1.6. Let $\hat{P}$ be the orbifold structure of a 3-dimensional polyhedron P. We say that the orbifold structure $\hat{P}$ is *orderable* if the sides of P can be ordered so that each side has no more than three edges which are either of order 2 or included in a side of higher index. $\hat{P}$ is trivalent if each side F has three or less number of edges of order two or edges belonging to sides of higher class than F.

2. Andreev's Theorem.

In 1970, E.M. Andreev provides a complete characterization of 3-dimensional hyperbolic polyhedral with finite volume having non-obtuse dihedral angles on his article [3]. Therefore Andreev's theorem is a fundamental tool for classification of 3-dimensional hyperbolic Coxeter polyhedrons. Some elementary faces about polyhedral are essential before we state Andreev's theorem.

Definition 2.1. A cell complex on S^2 is called *trivalent* if each vertex is the intersection of three faces. A 3-dimensional *combinatorial polyhedron* is a cell complex C on S^2 that satisfied the following condition:

(1) Every edges of C is the intersection of exactly two faces.
(2) A non-empty intersection of two faces is either an edge or a vertex.
(3) Every faces contains not fewer than 3 edges. If a face contains n edges then n is called the *length* of the face.

Suppose C^* be the dual complex of C in S^2 . Then C^* is a simplicial complex which embed in the same S^2 so that the vertex correspond to face of C, etc. A simple closed curve Γ in C^* is called *k-circuit* if it is formed by k edges of C^*. A k-circuit Γ is called *prismatic k-circuit* if the intersection of any two edges of C intersected by Γ is empty. If a prismatic k-circuit meets the edges $e_1, e_2, ..., e_k$ of C successively then we say that the edges $F_1, F_2, ..., F_k$ are an *k-prismatic element* of C.

Theorem 2.2 (Andreev, 1970), *Let C be an combinatorial polyhedron and suppose that non-obtuse angles* $0 < \alpha_{ij} \le \frac{\pi}{2}$ *are given corresponding to each edge* $F_{ij} = F_i \cap F_j$ *of C where* F_i *and* F_j *are the faces of C. Then there exist a convex hyperbolic polyhedron P with finite volume in 3-dimensional hyperbolic space which realize C with dihedral angles* α_{ij} *at the edge* F_{ij} *if and only if the following six conditions hold:*

(1) $0 < \alpha_{ij} \le \frac{\pi}{2}$

(2) *If* $F_{ijk} = F_i \cap F_j \cap F_k$ *is a vertex of C then* $\alpha_{ij} + \alpha_{jk} + \alpha_{ki} \ge \pi$. *If* F_{ijkl} *is a vertex then* $\alpha_{ij} + \alpha_{jk} + \alpha_{kl} + \alpha_{lk} = 2\pi$.

(3) *If* Γ *is a prismatic 3-circuit which intersects edges* F_{ij}, F_{jk}, F_{ki} *of C then* $\alpha_{ij} + \alpha_{jk} + \alpha_{ki} < \pi$

(4) *If* Γ *is a prismatic 4-circuit which intersects edges* $F_{ij}, F_{jk}, F_{ki}, F_{li}$ *of C then* $\alpha_{ij} + \alpha_{jk} + \alpha_{kl} + \alpha_{li} < 2\pi$

(5) *If C is a triangular prism with base* F_1, F_2 *, then*
$\alpha_{13} + \alpha_{14} + \alpha_{15} + \alpha_{23} + \alpha_{24} + \alpha_{25} < 3\pi$

(6) *If among the faces* F_i, F_j, F_k *we have* F_i *and* F_j *,* F_j *and* F_k*, but* F_i *and* F_k *not adjacent., but concurrent in one vertex and all three do not meet in one vertex, then* $\alpha_{ij} + \alpha_{jk} < \pi$.

Furthermore, this polyhedron is unique up to hyperbolic isometries. Since dihedral angles of Coxeter polyhedron is non-obtuse, Andreev's theorem provide a complete

characterization of 3-dimensional hyperbolic Coxeter polyhedron having more than four faces.
For tetrahedrons, only conditions (1) and (2) is sufficient as other conditions are trivially true.

3. Choi's Theorem.

Prof. Choi concentrated a class of Coxeter orbifolds which is called orderable Coxeter orbifolds and a certain type of orbifolds known as normal type orbifolds. In this class of orbifolds, we understand the restricted deformation space of orbifolds in real projective space from his article [6].

Definition 3.1. We denote *k(P)* the dimension of the projective group acting on a convex polyhedron *P*.

$$k(P)=\begin{cases} 3 \text{ if } P \text{ is a tetrahedron,} \\ 1 \text{ if } P \text{ is a cone with base} \\ \quad \text{a convex polygon which is no} \\ 0 \text{ otherwise} \end{cases}$$

Definition 3.2. A Coxeter group Γ is an abstract group define by a group presentation of form

$$\left(R_{i,};\left(R_iR_j\right)^{n_{ij}}\right), i, j \in I$$

Where I is a countable index set, $n_{ij} \in N$ is symmetric for i, j and $n_{ij} = 1$.

The fundamental group of the orbifold will be a Coxeter group with a presentation

$$R_i, i = 1, 2, \ldots f, \left(R_iR_j\right)^{n_{ij}} = 1$$

where R_i is associated with silvered sides and $R_{i,j}$ has order $n_{i,j}$ associated with the edge formed by the *i-th* and *j-th* side meeting.

A Coxeter orbifold whose polytope has a side *F* and a vertex *v* where all other sides are adjacent triangles to *F* and contains *v* and all edge orders of *F* are 2 is called a *cone-type* Coxeter orbifold. A Coxeter orbifold whose polyhedron is topologically a polygon times an interval and edges orders of top and bottom sides are 2 is called a *product-type* Coxeter orbifold. If $\hat{P}$ is not above type and has not finite fundamental group, then $\hat{P}$ is said to be a *normal-type* Coxeter orbifold.

Let e be the number of edges of *P* and e_2 be the number of edges of edge order 2. Let v be the number of vertices of *P* and f be the number of faces of *P*.

Theorem 3.3 (Choi, 2006). *Let P be a convex polyhedron and* $\hat{P}$ *be given a normal type Coxeter orbifold structure. Let k(P) be the dimension of the group of projective automorphisms acting on P. Suppose that* $\hat{P}$ *is orderable.* Then $\hat{P}$ is projectively deformable if and only if $3f - e - e_2 - k(P) > 0$.

4. Main Results

Now we are ready to establish the main results.

Proposition 4.1. *Let T be a 3-dimensional Coxeter tetrahedron and* $\hat{P}$ *be its Coxeter orbifold structure of P. If* $\hat{P}$ *is orderable and projectively deformable then* $e_2 < 3$.

Proof. By Choi's theorem 1.3.3,

$$3f - e - e_2 - k(T) > 0.$$

By definition 2.2.1, for the tetrahedron *T*, $k(T) = 3$. Since $f = 4, e = 6$, therefore

$$3f - e - e_2 - k(T) > 0$$
$$\Rightarrow 3\cdot 4 - 6 - e_2 - 3 > 0$$
$$\Rightarrow 3 > e_2$$

Proposition 4.2. *Let T be a 3-dimensional Coxeter tetrahedron with finite volume and* T *be its Coxeter orbifold structure of P. If* $\hat{P}$ *is orderable and projectively deformable then the order of the edges at each vertex is one of the form*

$$(3,3,3),(2,3,3),(2,3,4),(2,3,5),$$
$$(2,3,6),(2,4,4)(2,2,3)$$

Proof. Suppose there is an edge of order $n \geq 7$ at one vertex v_1. Suppose that r_1, r_2 be the order of the two edges at a vertex v_1.

By Andreev's first condition, we have

$$\frac{1}{n}+\frac{1}{r_1}+\frac{1}{r_2} \geq 1 \Rightarrow \frac{1}{r_1}+\frac{1}{r_2} \geq 1-\frac{1}{n} \geq \frac{6}{7} \qquad [n \geq 7]$$

Since $r_1, r_2 \geq 2$, therefore $\frac{1}{r_1}+\frac{1}{r_2} \leq 1$.

Hence $\frac{6}{7} \le \frac{1}{r_1} + \frac{1}{r_2} \le 1$.

If $r_1 \ge 3$ then $\frac{1}{r_1} + \frac{1}{r_2} \le \frac{1}{3} + \frac{1}{2} = \frac{5}{6} < \frac{6}{7}$.

This is a contradiction. Hence $r_1 = 2$.

If $r_2 \ge 3$ then $\frac{1}{r_1} + \frac{1}{r_2} \le \frac{1}{2} + \frac{1}{3} = \frac{5}{6} < \frac{6}{7}$.

This is a contradiction. Hence $r_2 = 2$.

Therefore the edge order at vertex v_1 is $(2,2,n), n \ge 7$.
Since the edge of order n is adjacent with two vertex of edge order of the form $(2,2,n), n \ge 7$, therefore the number of edges of order 2 is 4 and hence $e_2 \ge 4$.

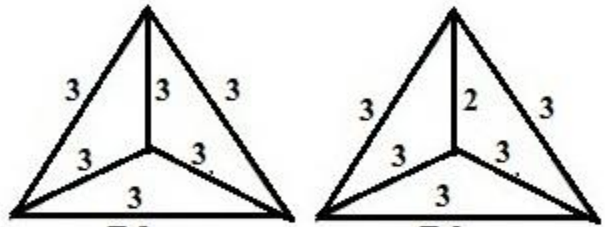
Since T is orderable and deformable, therefore by proposition 2.1, $e_2 < 3$. Therefore there is no vertex of edge order of the form $(2,2,n), n \ge 7$.
If the edge order at one vertex is of the form $(2,2,n), n \ge 4$ then there is one more edge of order 2 at the other end of the edge of order *n*. Then $e_2 \ge 3$. This is a contradiction.
By Andreev's 2nd condition, the edge order at each vertex is one of the form

$$(3,3,3),(2,3,3),(2,3,4),(2,3,5),$$
$$(2,3,6),(2,4,4)(2,2,3).$$

Theorem 4.3. *Let T be a hyperbolic Coxeter tetrahedron with finite volume and T be its Coxeter orbifold structure. Suppose that T is orderable and projectively deformable. Then the total number of Coxter hyperbolic orderable and deformable tetrahedrons T is 13 and these are T-1~13 as in figure 1~5.*
Proof. Since there are at most two edges of order 2, therefore $e_2 = 0$ or 1 or 2.

If $e_2 = 0$ then there are no vertices incident with edge of order 2. Therefore the edge order at all vertices is of the form $(3,3,3)$ by proposition 2.2

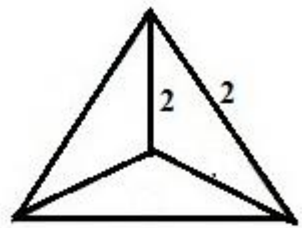
If $e_2 = 1$ then there are at most two vertices incident with edge of order 2. Then there are two vertices which don't incident with edge order 2. Therefore the edge order at these vertices is of the form $(3,3,3)$. Thus the tetrahedrons are as in figure 1:



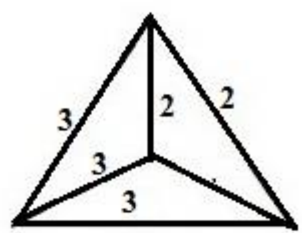
T-1 T-2

Figure-1

If $e_2 = 2$ then there are two edges of order 2. There are two possibilities: Both the edges of order 2 disjoint or adjacent.
If both the edges of order 2 are adjacent, then we assign the edge order 2 as follows:



There is one vertex which does not adjacent with an edge of order 2, therefore edge order at this vertex is $(3,3,3)$ as follows:



There is only one edge remain to assign order which can be 3,4,5,6. Therefore the tetrahedrons are as in figure-2.

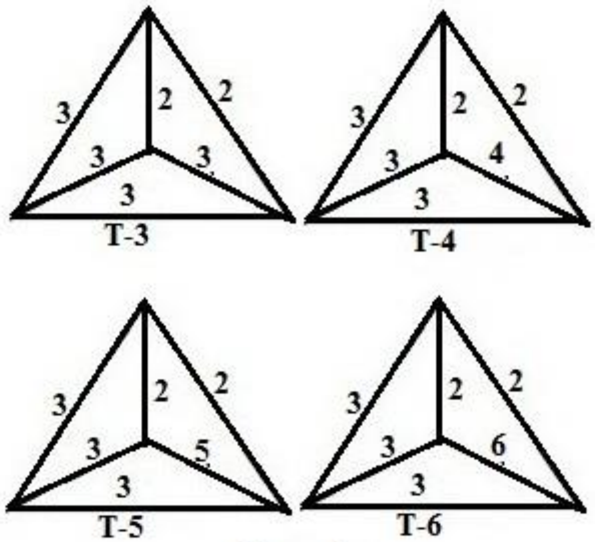
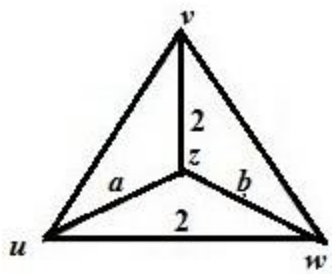


Figure-2

If both the edges of order 2 are disjoint, then we assign the edge order 2 as follows:



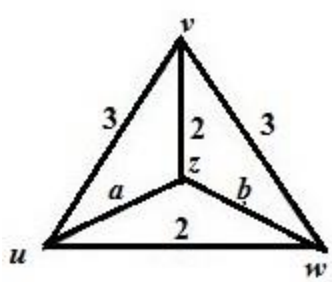
To avoid symmetries, we assume $a \le b$. Since each vertex is adjacent with exactly one edge of order 2, therefore the order of the edges at each vertex is one of the forms

$(2,3,3),(2,3,4),(2,3,5),(2,3,6),(2,4,4)$

Since the above figure is symmetric, first we assign the order of the edges at the vertex *v*.

We can divide the ordering into five cases.

Case-I: Suppose that the edge order of the edges at *v* be $(2,3,3)$ as follows:



Therefore (a,b) is one of the forms

$(3,3),(3,4),(3,5),(3,6),(4,4)$.

Then we have the new symmetries as in figure-3:

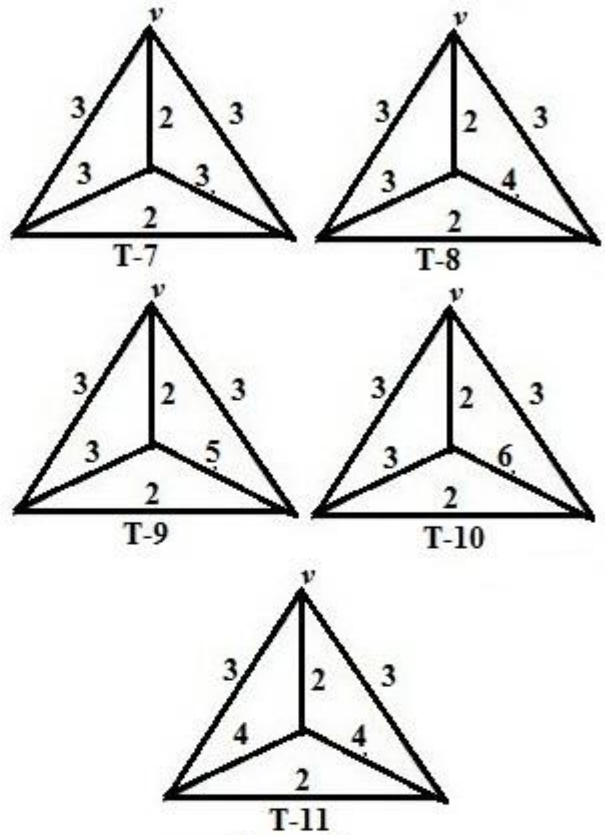
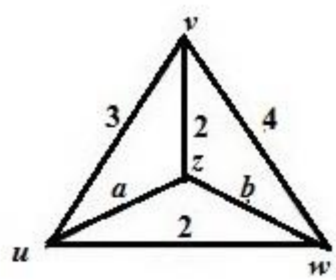


Figure-3

Case-II: Suppose that the edge order of the edges at *v* be $(2,3,4)$ as follows:

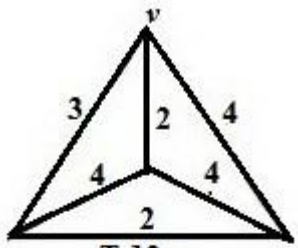


The order of the edges at each vertex is one of the forms $(2,3,3),(2,3,4),(2,3,5),(2,3,6),(2,4,4)$.

If $a = 3$ then this is same as case-I by symmetry $u \leftrightarrow v$.

Then $a \ge 4$ and hence $b \ge 4$.

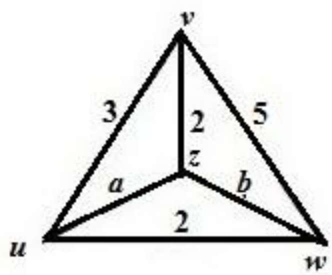
Therefore (a,b) is $(4,4)$. Then we have the new symmetry as in figure-4:



T-12

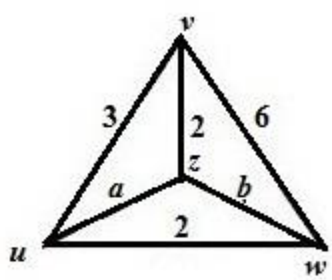
Figure-4

Case-III: Suppose that the edge order of the edges at *v* be $(2,3,5)$ as follows:



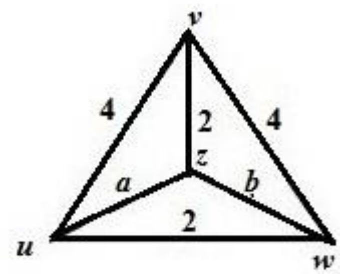
At the vertex *w*, the edge order is one of the forms $(2,3,3),(2,3,4),(2,3,5),(2,3,6),(2,4,4)$, therefore $b=3$ and hence $a=3$. Thus this is same as case-I by symmetry $u \leftrightarrow v$.

Case-IV: Suppose that the edge order of the edges at *v* be $(2,3,6)$ as follows:

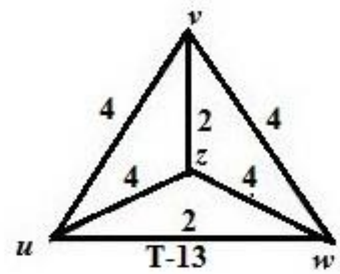


At the vertex *w*, the edge order is one of the forms $(2,3,3),(2,3,4),(2,3,5),(2,3,6),(2,4,4)$, therefore $b=3$ and hence $a=3$. Thus this is same as case-I by symmetry $u \leftrightarrow v$.

Case-V: Suppose that the edge order of the edges at *v* be $(2,4,4)$ as follows:



At the vertex *u*, the edge orders is one of the forms $(2,3,3),(2,3,4),(2,3,5),(2,3,6),(2,4,4)$, therefore $a=3$ or $a=4$. If $a=3$ then this is same as case-II by symmetry $u \leftrightarrow v$. Thus $a=4$ and hence $b=4$. We have the new symmetry as in figure-5:



T-13

Figure-5

Therefore there are total 13 orderable and deformable hyperbolic Coxeter tetrahedrons with finite volume and these are *T*-1~13 as in the figures 1~5.

5. Conclusions

In this article, we proved that the number of orderable and deformable hyperbolic Coxeter tetrahedrons with finite volume is exactly 13. It can be extended to find all the 3-dimensional orderable and deformable hyperbolic Coxeter polyhedrons with finite volume.

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