

Study of modified Casson's fluid model in modeled normal and stenotic capillary-tissue diffusion phenomena

Sapna Ratan Shah

Department of Mathematics
 Harcourt Butler Technological Institute, Kanpur- 208002,(India).
sapna1980jan@rediffmail.com

Abstract

The study focuses on the behavior of diffusion phenomenon in the normal and stenosed capillary-tissue exchange system where the rheology of flowing blood in the capillary is characterized by the generalized Casson's fluid model. Assessment of the severity of the disease could be made possible through the variation of a parameter named as retention parameter. The concentration profile and associated physiological diffusion variable involved in the study for normal and diseased state have been analyzed. The model is also employed to study the effect of shape of stenosis on flow characteristics. An extensive quantitative analysis is performed through numerical computations of the desired quantities having physiological relevance through their graphical representations so as to validate the applicability of the present model.

Keywords: Casson's fluid model, Capillary-tissue exchange, Resistance to flow, Wall shear stress, Stenosis shape parameter.

1. Introduction:

Stenosis is formed by substances depositing on vessel walls. A stenosis may lead to partial or total vessel blockage in some instances and therefore poses a serious medical problem, [1, 2]. The actual reason for formation of stenosis is not known, but its effect over the flow characteristics has been studied by many research workers, [3, 4, 5, 6]. Flow and diffusion through capillary-tissue exchange system has been identified as one of the thrust areas of research. In narrow capillaries, at times, the arterial transport become much larger as compared to axial transport and it contributes to the development of atherosclerotic plaques, greatly reducing the capillary diameter. The problem of flow and diffusion become much more difficult through a capillary with stenosis at some region. The response of blood flow through an artery under stenotic conditions has been attempted by [7, 8]. Accordingly, considerable effort has been expended

studying the fluid mechanics of flow through a stenosis [9, 10, 11]. Several workers [12, 13, 14] proposed various representative models for blood in narrow capillaries. Viscosity depending on the local variation of the concentration of the suspended cells has been introduced by [15, 16]. Perkkio and Keskinen [17] studied the effect of concentration on viscosity and the effect of the concentration on blood flow through a vessel with stenosis and found it an important aspect from physiological point of view. Kang and Erigen [18] have also discussed the effect of the variation of concentration of the suspended cells of blood. The theoretical study of Scott Blair and Spanner [19] pointed out that blood obeys the Casson's equation only in the limited range, except at very high and very low shear rate and that there is no difference between the Casson's plots and the Herschel-Bulkley plots of experimental data over the rang where the Casson's plot is valid. Also he suggested that the assumptions include in the Casson's equation are unsuitable for cow's blood and that the Herschel-Bulkley equation represents fairly closely what is occurring in the blood. Since the Herschel-Bulkley equation contains one more parameter than as compared to Casson's equation, it would be expected that more detailed information about blood properties could be obtained by the use of the Herschel-Bulkley equation. Furthermore, the Herschel-Bulkley equation is reduced to the mathematical models, which describes the behavior of Newtonian fluid, Bingham fluid and power law fluid by taking appropriate value of the parameters.

Casson's fluid model:

The Casson's relation is commonly written as:

$$\left. \begin{aligned} \tau^{1/2} &= \tau_0^{1/2} + (\mu)^{1/2} \left(-\frac{du}{dr}\right)^{1/2}, & \text{if } \tau \geq \tau_0 \\ \left(\frac{du}{dr}\right) &= 0 & \text{if } \tau < \tau_0 \end{aligned} \right\} \quad (1)$$

$$\text{where } \tau_0 = -\frac{dp}{dz} \frac{R_c}{2}$$

R_c is the radius of the plug-flow region, τ_0 is yield stress, τ is wall shear stress and μ denotes Casson's viscosity coefficient.

The stenosis studied in the present study is shown in figure [1(a), (1)b]. It is assumed that the stenosis develops in the capillary wall in an axially non-symmetric but radially symmetric manner and depends upon the axial distance z and the height of its growth.

$$\frac{R(z)}{R_0} = 1 - A[L_0^{(m-1)}(z-d) - (z-d)^m], \quad d \leq z \leq d+L_0$$

$$= 1, \quad \text{otherwise,} \quad (2)$$

Where the parameter $A = \frac{\delta}{R_0 L_0^m} \frac{m^{m/(m-1)}}{(m-1)}$

where δ denotes the maximum height of stenosis at $z = (d + L_0 / m^{m/(m-1)})$. The ratio of the stenosis height to the radius of the normal artery is much less than unity. $R(z)$ and R_0 are the radius of the artery with and without stenosis respectively. L_0 is the stenosis length, d represents the location of stenosis and m is stenosis shape parameter. In the case of $m \geq 2$, stenosis shape parameter indicates an axially symmetric stenosis.

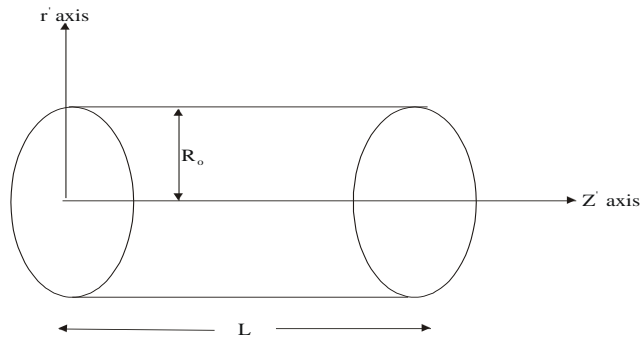


Figure 1 (a) Normal Artery

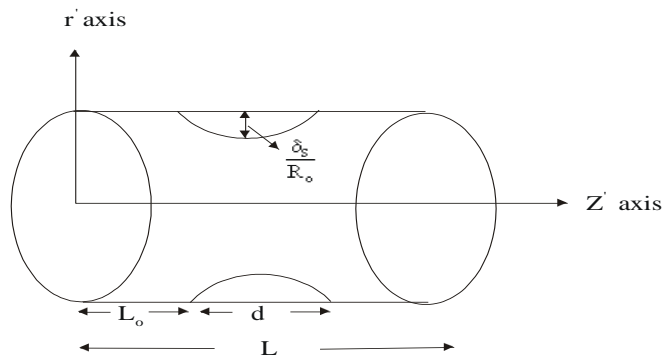


Figure 1 (b) Stenotic Artery

2. Formulation of the problem:

Following the report of Young [20], considering the axisymmetric laminar steady flow of blood, the general constitutive equation in the case of a mild stenosis subject to the additional conditions, may therefore be written as:

$$-\frac{\partial P}{\partial r} + \frac{1}{r} \frac{\partial (r \tau)}{\partial z} = 0,$$

$$-\frac{\partial P}{\partial r} = 0, \quad (3)$$

where p is the fluid pressure, $(-\partial p / \partial z)$ is pressure gradient in artery, where (z, r) are (axial, radial) coordinates with z measured along the axis and r measured normal to the axis of the artery, and (P, τ) are (Pressure, Shear stress).

The concentration equation for the solute is expressed by

$$u \frac{\partial C}{\partial z} = D \left(\frac{\partial^2 C}{\partial r^2} + \frac{1}{r} \frac{\partial C}{\partial r} \right) \quad (4)$$

Where C represents the concentration of the solute, u is the axial velocity and D the diffusion coefficient for the solute under consideration in the blood.

Boundary conditions: To solve the above system of equations, the following boundary conditions are introduced:

$$\frac{\partial u}{\partial r} = 0 \quad \text{at } r = 0$$

$$u = 0 \quad \text{at } r = R(z)$$

$$P = P_0 \quad \text{at } z = 0$$

$$P = P_L \quad \text{at } z = L$$

$$\frac{\partial C}{\partial r} = 0 \quad \text{at } r = 0$$

$$D \frac{\partial C}{\partial r} = VNC \quad \text{at } r = R \quad (5)$$

Where N is retention parameter, C is concentration; u is the axial velocity and D the diffusion coefficient.

3. Solution of the problem:

The Volume rate of flow using equation (3) is defined as,

$$Q = \pi \int_0^R r^2 \left(-\frac{du}{dr} \right) dr. \quad (6)$$

By integrating equation (6), using equations (3) and (5) we have,

$$Q = \frac{\pi R^4}{8\mu} \left(-\frac{dp}{dz}\right) \left[1 - \frac{16}{7} \left(\frac{R_c}{R}\right)^{1/2} + \frac{4}{3} \left(\frac{R_c}{R}\right) - \frac{1}{21} \left(\frac{R_c}{R}\right)^4\right], \quad (7)$$

Equation (7) can be rewritten as;

$$Q = \frac{\pi R^4}{8\mu} \left(-\frac{dp}{dz}\right) f(\bar{y}),$$

where

$$f(\bar{y}) = \left[1 - \frac{16}{7} (\bar{y})^{1/2} + \frac{4}{3} (\bar{y}) - \frac{1}{21} (\bar{y})^4\right],$$

$$\text{with } \bar{y} = \frac{R_c}{R} \ll 1.$$

From above equation pressure gradient is written as follows,

$$\left(-\frac{dp}{dz}\right) = \frac{8\mu Q}{\pi R^4 f(\bar{y})} \quad (8)$$

Integrating equation (8) using the condition $P = P_0$ at $z = 0$ and $P = P_L$ at $z = L$. We have,

$$\Delta P = P_L - P_0 = \frac{8\mu Q L}{\pi R_0^4} \int_0^L \frac{dz}{(R(z)/R_0)^4 f(\bar{y}(z))} \quad (9)$$

The resistance to flow (resistive impedance) is denoted by λ and defined as follows [Young, (4)],

$$\lambda = \frac{P_L - P_0}{Q} \quad (10)$$

The resistance to flow from equation (10) using equations (9) is written as,

$$\lambda = 1 - \frac{L_0}{L} + \frac{f_0}{L} \int_0^L \frac{dz}{(R(z)/R_0)^4 f(\bar{y}(z))} \quad (11)$$

where f_0 is given by

$$f_0 = \left[1 - \frac{16}{7} \left(\frac{R_c}{R_0}\right)^{1/2} + \frac{4}{3} \left(\frac{R_c}{R_0}\right) - \frac{1}{21} \left(\frac{R_c}{R_0}\right)^4\right].$$

Following the apparent viscosity (μ_{app}) is defined as follows;

$$\mu_{app} = \frac{1}{\left(R(z)/R_0\right)^4 f(\bar{y})} \quad (12)$$

The shearing stress at the wall can be defined as;

$$\tau_R = \left[\tau_0^{1/2} + \left(-\mu \frac{du}{dr}\right)_{r=R(z)}^{1/2}\right]^2 \quad (13)$$

To solve the Eq. (4) takes the form:

$$\frac{v R_0^2}{D_1 L} \frac{\partial C_1}{\partial x} = \frac{\partial^2 C_1}{\partial \eta^2} + \frac{1}{\eta} \frac{\partial C_1}{\partial \eta} \quad (14)$$

The boundary conditions are:

$$\begin{aligned} \frac{\partial C_1}{\partial \eta} &= 0 & \text{at } \eta &= 0, \\ D_1 \frac{\partial C_1}{\partial \eta} &= VNC_1 & \text{at } \eta &= \frac{R}{R_0} \end{aligned} \quad (15)$$

$$u = \frac{R_0^2}{4\mu L} \left(\frac{dp}{dx}\right) \left[R'^2 + 2R'^2 \eta - \frac{2}{5} \eta^3 R_c^3 - \frac{8}{3} \eta_c^2 \right] \quad (16)$$

On using Eq. (16) the solution for concentration subject to the boundary conditions is given as:

$$C_1 = \frac{R_0^3}{4\mu L^2 D_1} \left(-\frac{dp}{dx}\right) \left(\frac{\partial C_1}{\partial x}\right) \left[\left(\frac{R' \eta^3}{5}\right) - \left(\frac{\eta^4}{3}\right) - \left(\frac{R^5 \eta^2}{5} - \frac{54 \eta^{5/2}}{6}\right) + 2\eta_c^2 \left(\frac{R' \eta_c^2}{3} - \frac{R_c^2 \eta^2}{5}\right) - \left(\frac{\partial C_1}{\partial z} \frac{\bar{u}}{5 D_1 L^5} \eta_c^2\right) + M \right] \quad (17)$$

Where

$$M = \left[\frac{R_0^4}{4\mu L^2} \left(-\frac{dp}{dz} \right) \left\{ \begin{aligned} & \frac{R^3}{4} \left(1 - \frac{VNR_0 R'}{D_1(2n+1)} \right)^3 \\ & - \frac{8}{D_1^2 N} \eta_c^2 R \left(\frac{5}{\eta_c} - \frac{VNR_0 R'}{D_1} \right) \\ & + \frac{\eta_c}{R^3} \left(1 - \frac{V^2 R_0}{D_1} \right) - \frac{R' R_0^2}{L} \left(1 - \frac{V^2 NR_0 R'}{D_1} \right) \end{aligned} \right\} \right]$$

4. Results discussion:

In order to have estimate of the quantitative effects of various parameters involved in the analysis computer codes were developed and to evaluate the analytical results obtained for resistance to blood flow, concentration profile and associated physiological diffusion variables for normal and diseased system associated with stenosis due to the local deposition of lipids have been determine. The results are shown in Fig 2-9 by using the values of parameter based on experimental data in capillary.

Fig 2-4 shows the results for resistance to flow for different values of stenosis shape parameter, stenosis length, stenosis size and yield stress. Resistance to flow decreases as stenosis shape parameter increases and increases as stenosis size, stenosis length and yield stress increases. Resistance to flow increase as stenosis grows or radius of artery decreases. This referred to as Fahraeus-Lindquist effect in very thin tubes. The present results are therefore consistent with the observation of Haldar [5, 4, 15]. Fig 5-

7 shows the results for apparent viscosity for different values of stenosis shape parameter, stenosis length, stenosis size and yield stress. Apprent viscosity increases as stenosis size, stenosis length increases and yield stress increases and decreases as stenosis shape parameter increases and results are compared with [17, 19]. It is clear that apparent viscosity increases as stenosis grows. But the same is not true in the absence of stenosis. In capillary flow, the viscosity of blood flow found to vary with the radius of the capillary. The development of stenosis accelerates the velocity of plasma between the cells. This in turn increases the concentration of red cell and viscosity of blood in stenotic region, therefore increases.

Fig (8) shows the diffution of large and small molecular weight nutrients within the capillary region for different values of stenosis size. Large molecular weight nutrients within the capillary region face more resistance to diffuse into the tissue and therefore the cells of the the deeper region are deprived of getting sufficient nutrition. This result is consistent with result of Tandon et al. [15]. Fig (9) represents the effects of retention parameter (N) on concentration in blood flow capillary region. Increasing values of retention parameter described the increase in retention of solute within the blood flow in the capillary region. The value of retention parameter (N=1) implies the complete retention. No solute or fluid diffuses and as retention parameter decreases from 1 to 0.4 more solute diffuses, which in turns, decreases the solute concentration in the capillary region. The variation of the values of retention parameter in the stenotic region may also be associated with the type of plaques deposited on the walls: calcified, fibrous or fatty plaque.

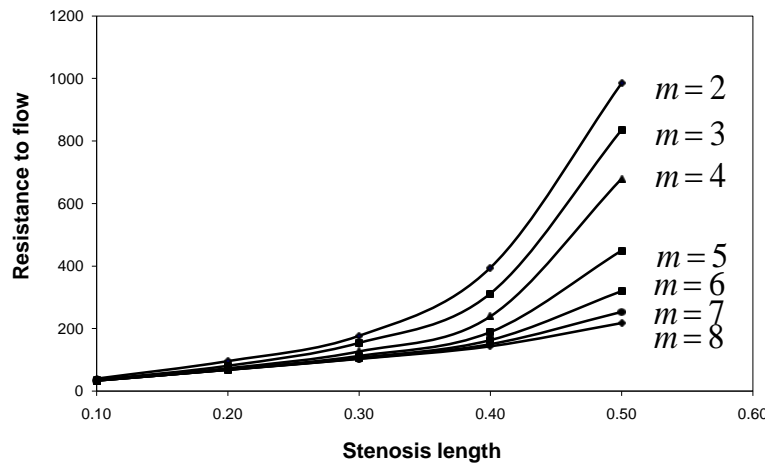


Fig (3) Variation of resistance to flow with stenosis length for different values of stenosis shape parameter

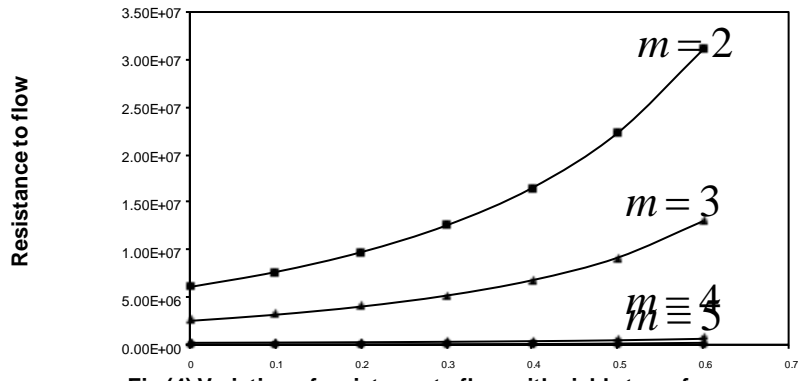


Fig (4) Variation of resistance to flow with yield stress for different values of stenosis shape parameter

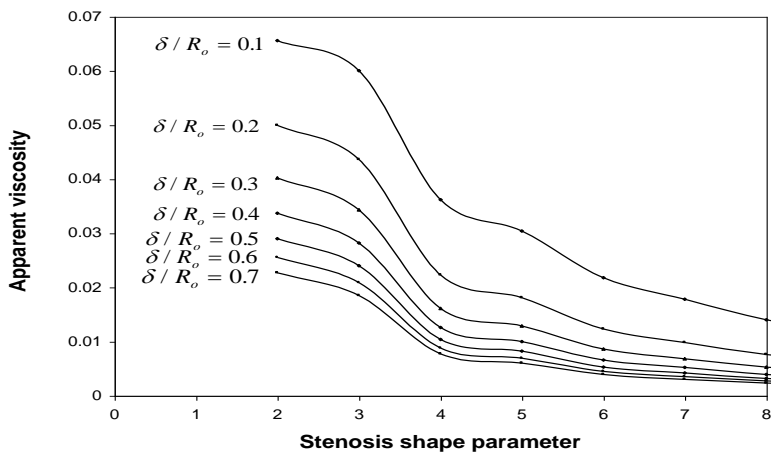


Fig (5) Variation of apparent viscosity with stenosis shape parameter for different values of stenosis size

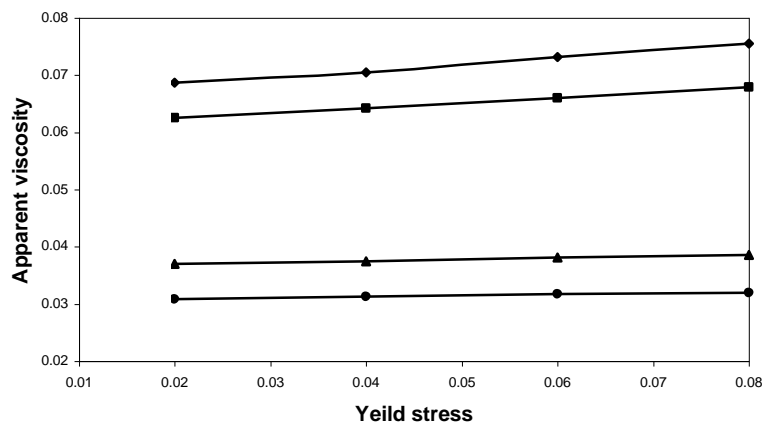


Fig (6) Variation of apparent viscosity with yield stress for different values os stenosis shape parameter

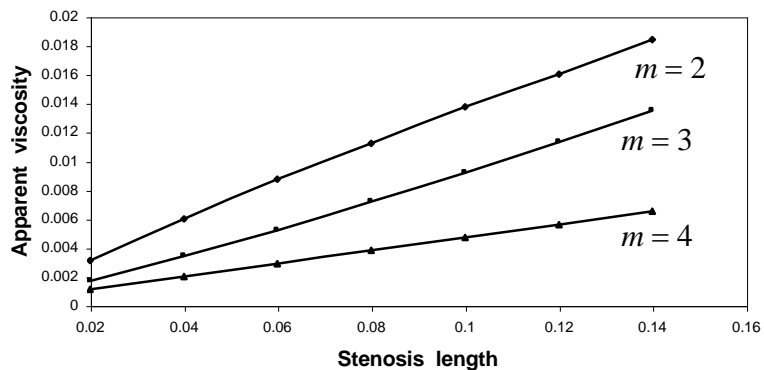


Fig (7) Variation of apparent viscosity with stenosis length for different values of stenosis shape parameter

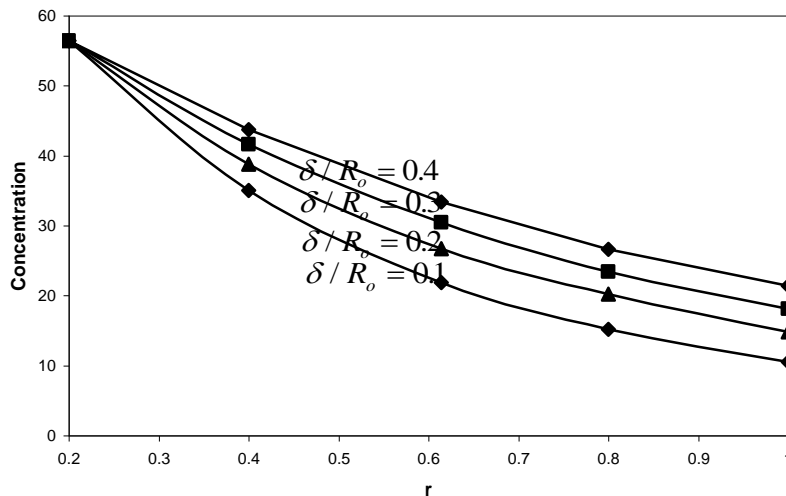


Fig (8) Concentration profile for different values of stenosis size

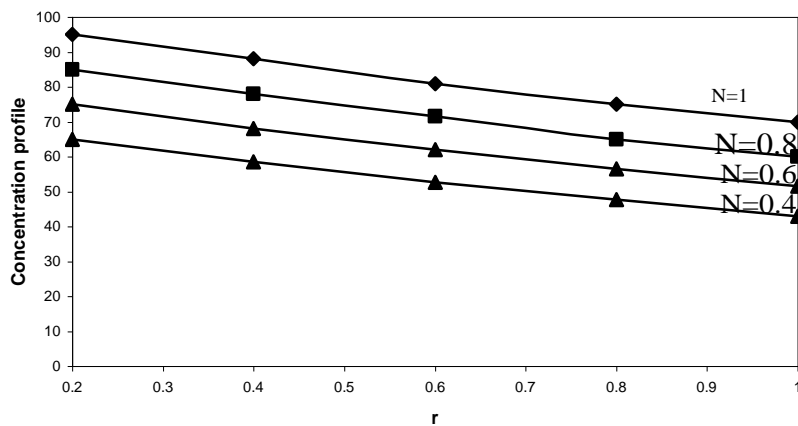


Fig (9). Concentration profile for different values of retention parameter (N)

5. Concluding remarks:

The present study incorporates the more realistic representation for blood in small diameter blood vessels and simultaneous dispersion of solute in capillary in normal and stenotic depending on various parameters including retention parameter. Casson's fluid model appears to be realistic in the sense that the equations are fairly closely to the blood flow and the central core region is easily represented and one more parameter index behavior (non-Newtonian nature of this fluid) is given in the model. The results are more encouraging and correlating well with the experimental observation that deeper region cells are deprived of the nutrients in the stenotic region. More experimental results are required for further development from clinical point of view.

References:

1. H. Mogens "An Evaluation of a Micropolar A. Harry Model for Blood Flow through an Idealized Stenosis", *J. Biomechanics*, Vol. 22, No.3. pp. 21-218. (1989)
2. A. Ogulu, J. Fiz, *Mal Vol.15*, pp.149. (1994)
3. J.B. Shukla, R.S. Parihar and B.R.P. Rao, "Effect of Stenosis on Non-newtonian Flow of Blood in an Artery". *Bull. Math. Biol.* Vol. 42, pp.283-294. (1980-a).
4. J.B. Shukla, R.S. Parihar and B.R.P. Rao. "Biorheological Aspects of Blood Flow through Artery with Mild Stenosis: Effects of Peripheral Layer." *Biorheology* vol. 17, pp.403-410. (1980-c).
5. K. Haldar, "Blood flow and red blood cell deformation, *Math. Biol.*Vol-47545. (1985)
6. S. Chakraborty and A. Chowdhury, *Gosh: Rheol. Acta* Vol. 27, pp.418. (1988)
7. T. Fukushima, T. Azuma and T. Matsuzawa: "Numerical Analysis of Blood Flow in the Vertebral Artery." *J. Biomech. Engng.* Vol. 104, pp.143-147. (1982).
8. D.F. Young: "Fluid Mechanics of Arterial Stenoses". *J. Biomech. Engng.* Vol-100, pp.157-175. (1979).
9. J.S. Lee and Y.C. Fung: "Flow in Locally Constricted Tubes at Low Reynolds Numbers". *J. App. Mech.*, Vol. 37, pp. 9-16. (1970).
10. V., cristini.& Ghassan, S. Kassab, *Computer Modeling of red blood cell Rheology in the Microcirculation: A Brief Overview.* A.M.B.E. 33 (2005) 1-4.
11. Sugihara- Seki, M. Motion of a sphere in a cylindrical tube filled with a Brinkman medium. *Fluid Dyn. Res.* 34 (2003) 59-76.
12. T.W Secomb Pries. AR, Blood flow and red blood cell deformation in non-uniform capillaries: effect of endothelial surface layer. *Microc.* 9, (2005) 189-196
13. SE SE, Kurland GS: Tube flow behavior and shear stress rate characteristics of canine blood: *Am. J. Physiol.* 203(1962)417-421.
14. AC Erigen: Simple microfluids, *Int. J. Eng. Sci.* 2(4) (1964) 205-217.
15. PN Tandon and Agrawal R: A study on nutritional transport in synovial joint, *Int. J. of Computer and Mathematics with applications*:17(7) (1989)1131-1141.
16. PN Tandon and Pal TS: On tranmural fluid exchange and variation of viscosity of blood flowing through permeable capillaries: *Med. & Life Sci. Eng.* 5(1)(1979)18-29.
17. J. Perkkio: "Locally Constricted Tubes at Low " *Bull. Math. Biol.* 49 (1983) 259-269.
18. KC Kang, Numerical Analysis of Blood Flow *Bull. Math. Biol.* 38 (1976) 135-142.
19. GW Scott GW and Spanner DC, An introduction to biorheology, Elsevier Scientific Publishing Company Amsterdam,.4,(1974).51.
20. DF Young and Tsai FY, Flow characteristics in models of arterial stenosis-I, steady flow, *J. Biomech.*, Vol. 6, pp. 395-410, (1973).